

A COMBINATORIAL ANALYSIS OF SEVERI DEGREES

FU LIU

ABSTRACT. Fomin and Mikhalkin give formulas for computing Severi degrees using long-edge graphs. We define τ -graphs that generalize their graphs, and a closely related family of combinatorial objects we call (τ, \mathbf{n}) -words. By considering their generating functions, we prove a linearity conjecture of Block, Colley and Kennedy, which gives a combinatorial proof of the result that the polynomials appearing in the formal logarithm of the generating function for Severi degrees are quadratic.

1. INTRODUCTION

The *Severi degree*, denoted by $N^{d,\delta}$, is the degree of the Severi variety. It counts the number of curves of degree d with δ nodes passing through $\frac{d(d+3)}{2} - \delta$ general points in the complex projective plane. If $d \geq \delta + 2$, the Severi degree $N^{d,\delta}$ coincides with the Gromov-Witten invariant $N_{d, \frac{(d-1)(d-2)}{2} - \delta}$, which counts maps from curves to the plane.

In 1994, Di Francesco and Itzykson [5] conjectured that for fixed δ , the Severi degree $N^{d,\delta}$ is given by a *node polynomial* $N_\delta(d)$ for sufficiently large d . Later, Göttsche [7, Conjecture 2.4] gave a stronger conjecture on the existence of universal polynomials enumerating curves on smooth projective surfaces, which is known as the Göttsche-Yau-Zaslow formula. Fomin and Mikhalkin [6, Theorem 5.1] established the polynomiality of $N^{d,\delta}$ using tropical geometry and floor decomposition. Recently, Tzeng [13] and Kool-Shende-Thomas [9] independently proved Göttsche's conjecture.

The *threshold* of the polynomiality of $N^{d,\delta}$ is the value $d^* = d^*(\delta)$ such that $N^{d,\delta} = N_\delta(d)$ for all $d \geq d^*$. Fomin and Mikhalkin [6] showed that $d^* \leq 2\delta$; Block [1] lowered it to $d^* \leq \delta$; and most recently Kleiman and Shende [8] proved the bound $d^* \leq \lceil \delta/2 \rceil + 1$ conjectured by Göttsche.

Node polynomials $N_\delta(d)$ have been computed up to $\delta = 14$ [1].

We consider the generating function for Severi degrees:

$$\mathcal{N}(d) = 1 + \sum_{\delta \geq 1} N^{d,\delta} t^\delta,$$

and its formal logarithm

$$\mathcal{Q}(d) = \log(\mathcal{N}(d)) = \sum_{\delta \geq 1} Q^{d,\delta} t^\delta.$$

We have the following relations between $N^{d,\delta}$ and $Q^{d,\delta}$: for $\delta \geq 1$,

$$(1.1) \quad N^{d,\delta} = \sum_{i \geq 1} \frac{1}{i!} \sum_{(\delta_1, \dots, \delta_i)} \left(\prod_{j=1}^i Q^{d,\delta_j} \right),$$

The author is partially supported by the Collaboration Grants for Mathematicians by Simons foundation.

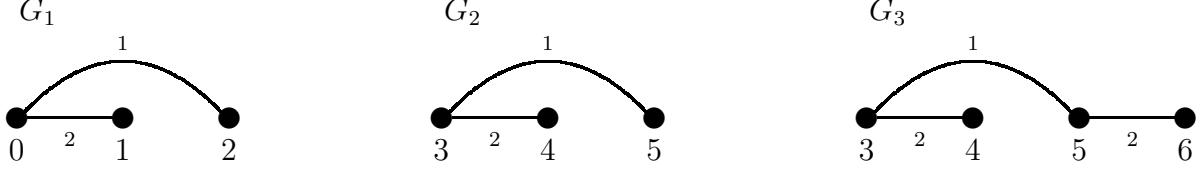


FIGURE 1. Examples of long-edge graphs

$$(1.2) \quad Q^{d,\delta} = \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(\delta_1, \dots, \delta_i)} \left(\prod_{j=1}^i N^{d, \delta_j} \right),$$

the second summations in both equations are over i -composition $(\delta_1, \dots, \delta_i)$ of δ , i.e., $\delta_1, \dots, \delta_i$ are positive integers summing up to δ .

Assume $d \geq \lceil \delta/2 \rceil + 1$. Then for any i -composition $(\delta_1, \dots, \delta_i)$ of δ , we have $d \geq \lceil \delta_j/2 \rceil + 1 \geq d^*(\delta_j)$ for each j . Hence, we know that $N^{d, \delta_j} = N_{\delta_j}(d)$ for each j . Therefore, (1.2) becomes

$$Q^{d,\delta} = \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(\delta_1, \dots, \delta_i)} \left(\prod_{j=1}^i N_{\delta_j}(d) \right).$$

Hence, $Q^{d,\delta}$ is a polynomial in d when $d \geq \lceil \delta/2 \rceil + 1$. We denote this polynomial by $Q_\delta(d)$. Although the degree of $N_\delta(d)$ was shown to be 2δ , the polynomial $Q_\delta(d)$, which is an alternating sum of $N_\delta(d)$'s, turns out to be quadratic. This result follows from the Göttsche-Yau-Zaslow formula proved by Tzeng and Kool-Shende-Thomas. (See Proposition 3.1 in [10].) In this paper, we will provide another proof of the quadraticity of $Q_\delta(d)$ by proving a certain function associated to *long-edge graphs* is linear. We give a brief introduction to the objects in our results below, and will fill in the details in Section 2.

Brugallé and Mikhalkin [4, 3] introduced “(marked) labeled floor diagrams” and gave an enumerative formula for the Severi degree $N^{d,\delta}$ in terms of these diagrams. Fomin and Mikhalkin [6] reformulated Brugallé and Mikhalkin’s results by introducing a “template decomposition” of labeled floor diagrams. They first constructed a bijection between labeled floor diagrams and *long-edge graphs* and then gave a natural decomposition of long-edge graphs into *templates*. (Fomin and Mikhalkin did not name the graphs they use; the terminology “long-edge graphs” was first introduced in [2].)

Definition 1.1. A *long-edge graph* G is a graph (V, E) with a weight function ρ satisfying the following conditions:

- a) The vertex set $V = \mathbb{N} = \{0, 1, 2, \dots\}$, and the edge set E is finite.
- b) Multiple edges are allowed, but loops are not.
- c) The weight function $\rho : E \rightarrow \mathbb{P}$ assigns a positive integer to each edge.
- d) There are no *short edge*, i.e., there’s no edge connecting i and $i + 1$ with weight 1.

We often draw the vertices $0, 1, 2, \dots$ of long-edge graphs from left to right and label each edge with its weight. Since all but finitely many vertices do not have incident edges, we often omit most of irrelevant vertices when we draw long-edge graphs. See Figure 1 for three examples of long-edge graphs.

Fomin and Mikhalkin associate to each long-edge graph a statistic $P(G)$, and then give an enumerative formula for computing the Severi degree $N^{d,\delta}$ in terms of long-edge graphs

using this statistic. We consider a logarithmic version of the statistic $P(G)$. For any long-edge graph G , we define

$$(1.3) \quad \Phi(G) := \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(G_1, \dots, G_i)} \left(\prod_{j=1}^i P(G_j) \right),$$

where the summation is over all the partitions of G .

Another concept we need is *shifted graphs*. For any long-edge graph G , we denote by $G_{(k)}$ the graph obtained by shifting all edges of G to the right k units, i.e., a weighted edge $\{i, j\}$ in G becomes a weighted edge $\{i+k, j+k\}$ in $G_{(k)}$. For instance, in Figure 1, graph G_2 is obtained by shifting graph G_1 . More precisely, $G_2 = (G_1)_{(3)}$.

Below is the main result of this paper.

Theorem 1.2. *Suppose G is a long-edge graph. Then $\Phi(G_{(k)})$ is a linear function in k for sufficiently large k .*

In 2012, the above theorem was conjectured by Block, Colley and Kennedy. They have since given in [2] an independent proof of their conjecture. The proof presented in this paper is more combinatorial and provides combinatorial objects to compute the coefficients of the linear function described in the theorem. We hope that it will provide some insight on how to compute the polynomials $Q_\delta(d)$. Since the node polynomials $N_\delta(d)$ and the polynomials $Q_\delta(d)$ are related in the same way as $N^{d,\delta}$ and $Q^{d,\delta}$ shown in (1.1) and (1.2), our method could provide good methods for computing $N^{d,\delta}$ as well.

We then recover the result on the quadraticity of $Q^{d,\delta}$ or $Q_\delta(d)$.

Corollary 1.3. *For any fixed δ , $Q^{d,\delta}$ is a quadratic polynomial in d for sufficiently large d .*

This paper is organized as follows. In Section 2, we give detailed description of elements in Fomin-Mikhalkin's formula for Severi degree using long-edge graphs, leading to the definition of $\Phi(G)$. In section 3, we prove Corollary 1.3 using our results, and give examples of how we can use the linear function $\Phi(G_{(k)})$ to compute the quadratic polynomial $Q_\delta(d)$. The proof of Theorem 1.2 consists of several reduction steps appearing in Sections 4, 5, 6 and 7. In section 4, we introduce τ -graphs, a generalization of long-edge graphs, and state our theorem in terms of τ -graphs. We then reduce the problem to proving a theorem on the generating function on τ -graphs (Theorem 4.11). In Section 5, we introduce another combinatorial object: (τ, \mathbf{n}) -words, a special family of which, denoted by $S_\tau(\mathbf{n}, t; \ell)$, has a reciprocity connection to τ -graphs. Using this connection, we reduce our problem (of proving Theorem 4.11) to proving a result on the generating function of $S_\tau(\mathbf{n}, t; \ell)$ (Proposition 5.9). In Section 6, we introduce a height function and a concept of irreducibility for (τ, \mathbf{n}) -words, and Proposition 5.9 is reduced to Lemma 6.7 which states that there is a unique way of decomposing words in $S_\tau(\mathbf{n}, t; \ell)$ into certain $t+1$ words. In Section 7, using the height function, we describe an algorithm to decompose (τ, \mathbf{n}) -words, which provides us a proof for Lemma 6.7, thus completing our proof of Theorem 1.2. In Section 8, we give examples of how to compute the linear function described in Theorem 1.2 through (τ, \mathbf{n}) -words. In particular, we provide an explicit formula (in Lemma 8.3) for the linear function when the long-edge graph only has one type of edges.

2. SEVERI DEGREES VIA LONG-EDGE GRAPHS

In this section, we state the Fomin-Mikhalkin's formula for computing Severi degree $N^{d,\delta}$ using long-edge graphs, preceded by all relevant definitions. Taking the logarithm of the generating function of their formula, we give a formula for $Q^{d,\delta}$ by defining functions Φ_d and Φ for long-edge graphs. We then state a more detailed version of Theorem 1.2 (Theorem 2.12) and an important result on the function Φ_d (Lemma 2.15).

Definition 2.1. Given a long-edge graph $G = (V, E)$ equipped with weight function ρ , we define the *multiplicity* of G to be

$$\mu(G) = \prod_{e \in E} (\rho(e))^2,$$

and the *cogenus* of G to be

$$\delta(G) = \sum_{e \in E} (l(e)\rho(e) - 1),$$

where for any $e = \{i, j\} \in E$ with $i < j$, we define $l(e) = j - i$.

We also define the *length* of G , denoted by $l(G)$, to be the largest vertex that has nonzero-degree. (Hence, any vertex $> l(G)$ is not incident to any edge.)

Note that any non-empty long-edge graph has positive cogenus.

Example 2.2. Consider G_1 and G_2 in Figure 1. As we discussed before, $G_2 = (G_1)_{(3)}$. It is clear that

$$\mu(G_1) = \mu(G_2) = 2^2 \cdot 1^1 = 4, \quad \delta(G_1) = \delta(G_2) = (2 \cdot 2 - 1) + (1 \cdot 2 - 1) = 2.$$

However, note that by our definition, $l(G_1) = 2$ but $l(G_2) = 5$.

Definition 2.3. Given a long-edge graph G , we say a tuple (G_1, \dots, G_i) of (non-empty) long-edge graphs is a *partition* of G if the disjoint union of the (weighted) edge sets of G_1, \dots, G_i is the (weighted) edge set of G .

By the definitions of multiplicity and cogenus, one checks that for any partition (G_1, \dots, G_i) of G , we have

$$(2.1) \quad \mu(G) = \prod_{j=1}^i \mu(G_j) \quad \text{and} \quad \delta(G) = \sum_{j=1}^i \delta(G_j).$$

Definition 2.4. Let G be a long-edge graph with associated weight function ρ . We define

$$\lambda_j(G) = \text{sum of the weights of edges } \{i, k\} \text{ with } i < j \leq k, \quad \forall j.$$

We say G is *allowable* if $j - 1 \geq \lambda_j(G)$ for each j .

Let d be a positive integer. A long-edge graph G is *allowable for d* if it satisfies the following conditions:

- a) G is allowable.
- b) $l(G) \leq d + 1$.
- c) Any edge that is incident to the vertex $d + 1$ has weight 1.

Example 2.5. Consider G_1 in Figure 1. We have

$$\lambda_1(G_1) = 3, \quad \lambda_2(G_1) = 1 \quad \text{and for any } j \geq 3, \quad \lambda_j(G_1) = 0.$$

Hence, G_1 is not allowable; thus is not allowable for any d .

Consider G_2 in Figure 1. We have

$$\lambda_4(G_2) = 3, \quad \lambda_5(G_2) = 1 \quad \text{and for any } j \neq 4, 5, \quad \lambda_j(G_2) = 0.$$

Hence, G_2 is allowable. Furthermore, one can check that G_2 is allowable for d if and only if $d \geq 4$.

Finally, one checks that G_3 in Figure 1 is allowable, and is allowable for d if and only if $d \geq 6$.

In the above example, $G_2 = (G_1)_{(3)}$ and is allowable. It is not hard to see that $(G_1)_{(k)}$ is allowable if and only if $k \geq 3$. We can find such a number for any long-edge graph: observe that given a long-edge graph G , its shift $G_{(k)}$ is allowable if and only if

$$k + j - 1 \geq \lambda_{k+j}(G_{(k)}) = \lambda_j(G), \quad \forall j.$$

Hence, it's natural to define

$$(2.2) \quad k_{\min}(G) := \max(0, \max\{\lambda_j(G) - j + 1\}).$$

Then we have the following result.

Lemma 2.6. $G_{(k)}$ is allowable if and only if $k \geq k_{\min}(G)$.

Definition 2.7. Suppose G is allowable. Let $\ell \geq l(G)$ be fixed. We create $\text{ext}(G)$ by adding $(j-1) - \lambda_j(G)$ (unweighted) edges connecting $j-1$ and j for each $j \leq \ell$.

An *extended ordering* of G is a total ordering of the vertices and edges of $\text{ext}(G)$ satisfying the following conditions:

- a) The ordering extends the natural ordering of the vertices $0, 1, 2, \dots$ of $\text{ext}(G)$.
- b) For any edge $e = \{a, b\}$, its position in the total ordering has to be between a and b .

We consider two extended orderings o and o' to be equivalent if there is an automorphism σ on the edges of $\text{ext}(G)$ such that

- a) If $\sigma(e) = e'$, then e and e' have the same vertices, and either have the same weights or are both unweighted.
- b) When applying σ on the ordering o , one obtains the ordering o' .

For any long-edge graph G , we define

$$P(G) = \text{the number of extended orderings (up to equivalence) of } G,$$

where by convention $P(G) = 0$ if G is not allowable, and then define

$$P_d(G) = \begin{cases} P(G) & \text{if } G \text{ is allowable for } d; \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.8. We remark that the definition of $P(G)$ is independent from the choice of ℓ as long as $\ell \geq l(G)$, because for any $j > l(G)$, the only edges that would be placed between vertices $j-1$ and j in an ordering are new edges that were added in creating $\text{ext}(G)$.

Example 2.9. Let G be the long-edge graph with only two edges of weight 2 connecting vertices 0 and 1. Then $G_{(k)}$ has two edges of weight 2 connecting vertices k and $k + 1$, and is allowable if and only if $k \geq 4$. Thus, $P(G_{(k)}) = 0$ for $k < 4$. For $k \geq 4$, in order to create $\text{ext}(G_{(k)})$, we need to add $k - 4$ unweighted edges connecting vertices k and $k + 1$. The number of extended orderings of $G_{(k)}$ only depends on how we order these $k - 4$ new edges and the original two edges in $G_{(k)}$. It is easy to see that $P(G_{(k)}) = \binom{k-4+2}{2} = \binom{k-2}{2}$ for $k \geq 4$.

Let G' be the long-edge graph with only one edge of weight 2 connecting vertices 0 and 1. By a similar discussion, we get that $P(G'_{(k)})$ is $k - 1$ for $k \geq 2$, and is 0 for $k < 2$.

From the above example, we observe that $P(G_{(k)})$ is a polynomial in k for $k \geq k_{\min}(G)$. This is not a coincidence.

Lemma 2.10 ([6], Lemma 5.8). *Suppose G is a long-edge graph. Then for $k \geq k_{\min}(G)$, the values $P(G_{(k)})$ are given by a polynomial in k whose degree is the number of edges in G .*

Now we describe Fomin-Mikhalkin's formula for the Severi degree using long-edge graphs.

Theorem 2.11 (Fomin-Mikhalkin). *The Severi degree $N^{d,\delta}$ is given by*

$$(2.3) \quad N^{d,\delta} = \sum_G \mu(G) P_d(G),$$

where the summation is over all the long-edge graphs of cogenus δ .

Applying the above theorem to the generating function $\mathcal{N}(d)$, we get

$$\mathcal{N}(d) = 1 + \sum_{\delta \geq 1} N^{d,\delta} t^\delta = 1 + \sum_G \mu(G) P_d(G) t^{\delta(G)},$$

where the summation is over all the (non-empty) long-edge graphs. Taking logarithms on both sides of the above formula, we obtain

$$(2.4) \quad Q^{d,\delta} = \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(G_1, \dots, G_i)} \left(\prod_{j=1}^i \mu(G_j) P_d(G_j) \right),$$

where the summation is over all the tuples (G_1, \dots, G_i) of (non-empty) long-edge graphs satisfying $\sum_{j=1}^i \delta(G_j) = \delta$. Since we can consider any such tuple a partition of a long-edge graph of cogenus δ , it is natural to give the following definition: for any long-edge graph G , we define

$$(2.5) \quad \Phi_d(G) := \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(G_1, \dots, G_i)} \left(\prod_{j=1}^i P_d(G_j) \right),$$

where the summation is over all the partitions of G .

With this definition and by (2.1), we can rewrite (2.4):

$$(2.6) \quad Q^{d,\delta} = \sum_G \mu(G) \Phi_d(G),$$

where the summation is over all the long-edge graphs of cogenus δ .

Recall the definition of $\Phi(G)$ given in the introduction:

$$(2.7) \quad \Phi(G) := \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(G_1, \dots, G_i)} \left(\prod_{j=1}^i P(G_j) \right).$$

It is easy to see that $\Phi_d(G) = \Phi(G)$ for sufficiently large d . (See Lemma 3.2 for a detailed statement of this fact.) Therefore, $\Phi(G)$ is an important statistic to consider.

We now state a more specific version of our main result Theorem 1.2.

Theorem 2.12. *Suppose G is a long-edge graph. Then $\Phi(G_{(k)})$ is a linear function in k for $k \geq k_{\min}(G)$.*

Example 2.13. Let G and G' be as described in Example 2.9. There are two partitions of $G_{(k)}$: $(G_{(k)})$ and $(G'_{(k)}, G'_{(k)})$. Hence,

$$\Phi(G_{(k)}) = \frac{(-1)^{1+1}}{1} P(G_{(k)}) + \frac{(-1)^{2+1}}{2} P(G'_{(k)}) P(G'_{(k)}) = P(G_{(k)}) - \frac{1}{2} P^2(G'_{(k)}).$$

Recall that we have computed $P(G_{(k)})$ and $P(G'_{(k)})$ in Example 2.9. Hence,

$$\Phi(G_{(k)}) = \begin{cases} 0 - \frac{1}{2} \cdot 0^2 = 0, & k = 0, 1; \\ 0 - \frac{1}{2} \cdot 1^2 = -\frac{1}{2}, & k = 2; \\ 0 - \frac{1}{2} \cdot 2^2 = -2, & k = 3; \\ \binom{k-2}{2} - \frac{1}{2} \cdot (k-1)^2 = -\frac{1}{2}(3k-5), & k \geq 4. \end{cases}$$

We see from the above example that unlike $P(G)$, the value of $\Phi(G)$ is not necessarily 0 when G is not allowable.

We come back to Equation (2.6) for computing $Q^{d,\delta}$. One benefit of computing $Q^{d,\delta}$ instead of $N^{d,\delta}$ is that a lot of terms in (2.6) vanish.

Definition 2.14. A long-edge graph Γ is a *template* if for any vertex $i : 1 \leq i \leq l(\Gamma) - 1$, there exists at least one edge $\{j, k\}$ satisfying $j < i < k$.

We say a long-edge graph G is a *shifted template* if G can be obtained by shifting a template; that is, if $G = \Gamma_{(k)}$ for some template Γ and some nonnegative integer k .

Lemma 2.15. *Suppose G is not a shifted template. Then*

$$\Phi_d(G) = 0.$$

Example 2.16. Consider again the three graphs in Figure 1. The graph G_1 is a template (and also a shifted template), the graph G_2 is a shifted template, and the graph G_3 is not a shifted template.

By Lemma 2.15, we have that $\Phi_d(G_3) = 0$.

We will include the proof of Lemma 2.15 in Section 4. We have the following corollary to Lemma 2.15.

Corollary 2.17.

$$(2.8) \quad Q^{d,\delta} = \sum_{\Gamma} \mu(\Gamma) \sum_k \Phi_d(\Gamma_{(k)}),$$

where the first summation is over all the templates of cogenus δ .

It is an easy fact that for any fixed δ , there are finitely many templates of cogenus δ . Hence, the first summation in (2.8) is finite. It is not hard to see that the second summation in (2.8) is finite as well. Therefore, there are finitely many terms in (2.8). Thus, one can use (2.8) to compute $Q^{d,\delta}$ and $Q_\delta(d)$.

3. COMPUTING $Q^{d,\delta}$ AND $Q_\delta(d)$

In this section, assuming Corollary 2.17 and Theorem 2.12, we prove Corollary 1.3. (Recall that Corollary 1.3 states that for fixed δ , the function $Q^{d,\delta}$ is quadratic in d for sufficiently large d .) We will then demonstrate how one can compute the quadratic polynomial $Q_\delta(d)$ from our results. The material presented in this section is irrelevant to the rest of the paper. The reader should feel free to skip it.

Because of (2.8), it is natural to define for each template Γ ,

$$Q^{d,\Gamma} = \mu(\Gamma) \sum_k \Phi_d(\Gamma_{(k)}).$$

Hence, it is sufficient to show that $Q^{d,\Gamma}$ is quadratic for sufficiently large d . We will prove this by analyzing functions Φ_d and Φ further and give a more precise formula for $Q^{d,\Gamma}$.

We start with a preliminary definition and a lemma.

Definition 3.1. Let G be a long-edge graph. We define

$$(3.1) \quad \epsilon(G) = \begin{cases} 1, & \text{if all edges adjacent to the vertex } l(G) \text{ have weight } 1; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.2. (i) $P_d(G) = P(G)$ if $d \geq l(G) - \epsilon(G)$ and is 0 if $d < l(G) - \epsilon(G)$.
(ii) $\Phi_d(G) = \Phi(G)$ if $d \geq l(G) - \epsilon(G)$ and is 0 if $d < l(G) - \epsilon(G)$.

Proof. If $d < l(G) - \epsilon(G)$, the graph G is not allowable for d ; if $d \geq l(G) - \epsilon(G)$, then G is allowable for d if and only if G is allowable. Hence, (i) follows.

We use (i) to prove (ii). Comparing equations (2.5) and (2.7) for $\Phi_d(G)$ and $\Phi(G)$, it is sufficient to prove that for any partition (G_1, \dots, G_i) of G , we have

$$\prod_{j=1}^i P_d(G_j) = \begin{cases} \prod_{j=1}^i P(G) & \text{if } d \geq l(G) - \epsilon(G); \\ 0 & \text{if } d < l(G) - \epsilon(G). \end{cases}$$

Suppose (G_1, \dots, G_i) is a partition of G . We check that for any $1 \leq j \leq i$, we have

$$l(G_j) - \epsilon(G_j) \leq l(G) - \epsilon(G).$$

and the equality holds for at least one j . Hence, if $d \geq l(G) - \epsilon(G)$, we have $d \geq l(G_j) - \epsilon(G_j)$ for all j . Thus, by (i), $\prod_{j=1}^i P_d(G_j) = \prod_{j=1}^i P(G)$. If $d < l(G) - \epsilon(G)$, since there exists j such that $l(G_j) - \epsilon(G_j) = l(G) - \epsilon(G) > d$, which implies $P_d(G_j) = 0$ by (i), we have $\prod_{j=1}^i P_d(G_j) = 0$. \square

Corollary 3.3. Suppose Γ is a template. Then

$$Q^{d,\Gamma} = \begin{cases} \mu(\Gamma) \sum_{k=1}^{d+\epsilon(\Gamma)-l(\Gamma)} \Phi(\Gamma_{(k)}), & \text{if } d \geq l(\Gamma) - \epsilon(\Gamma); \\ 0, & \text{if } d < l(\Gamma) - \epsilon(\Gamma). \end{cases}$$

Furthermore, $Q^{d,\Gamma}$ is a quadratic polynomial in d for $d \geq k_{\min}(\Gamma) + l(\Gamma) - \epsilon(\Gamma)$.

Proof. The first part follows immediately from Lemma 3.2/(ii) and the fact that $\Gamma_{(0)} = \Gamma$ is not allowable.

Suppose $d \geq k_{\min}(\Gamma) + l(\Gamma) - \epsilon(\Gamma)$. Then

$$(3.2) \quad Q^{d,\Gamma} = \mu(\Gamma) \left(\sum_{k=1}^{k_{\min}(\Gamma)-1} \Phi(\Gamma_{(k)}) + \sum_{k=k_{\min}(\Gamma)}^{d+\epsilon(\Gamma)-l(\Gamma)} \Phi(\Gamma_{(k)}) \right).$$

Clearly the first summation is a constant, and by Theorem 2.12, the second summation becomes a quadratic polynomial in d . \square

Remark 3.4. Corollary 3.3 not only implies Corollary 1.3, also tells us that $Q^{d,\delta}$ is quadratic in d for d satisfying $d \geq k_{\min}(\Gamma) + l(\Gamma) - \epsilon(\Gamma)$ for each template Γ of cogenus δ .

We will denote by $Q_{\Gamma}(d)$ the quadratic polynomial corresponding to $Q^{d,\Gamma}$. Therefore, we have

$$Q_{\delta}(d) = \sum_{\Gamma} Q_{\Gamma}(d).$$

where the summation is over all the templates of cogenus δ . In order to compute the polynomial $Q_{\delta}(d)$, we need to compute $Q_{\Gamma}(d)$ for each template Γ of cogenus δ . Figure 2 lists all the templates of cogenus 1 or 2. (The data in Figure 2 is mostly copied from Figure 10 in [6] except the last column.) Note that since every template Γ has at least one edge incident to the vertex 0, we omit labels of vertices when drawing a template Γ and assume that the vertices are $0, 1, \dots, l(\Gamma)$.






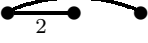



Γ	$\delta(\Gamma)$	$l(\Gamma)$	$\mu(\Gamma)$	$\epsilon(\Gamma)$	$\lambda(\Gamma)$	$k_{\min}(\Gamma)$	$P(\Gamma_{(k)})$ $k \geq k_{\min}(\Gamma)$	$\Phi(\Gamma_{(k)})$ $k \geq k_{\min}(\Gamma)$
	1	1	4	0	(2)	2	$k-1$	$k-1$
	1	2	1	1	(1,1)	1	$2k+1$	$2k+1$
	2	1	9	0	(3)	3	$k-2$	$k-2$
	2	1	16	0	(4)	4	$\binom{k-2}{2}$	$-\frac{1}{2}(3k-5)$
	2	2	1	1	(2,2)	2	$\binom{2k}{2}$	$-\frac{1}{2}(6k+1)$
	2	2	4	1	(3,1)	3	$2k(k-2)$	$-3k+1$
	2	2	4	0	(1,3)	2	$2k(k-1)$	$-3k$
	2	3	1	1	(1,1,1)	1	$3(k+1)$	$3k+3$
	2	3	1	1	(1,2,1)	1	$k(4k+5)$	$-3k-3$

FIGURE 2. The templates with $\delta(\Gamma) \leq 2$.

Below we first give an example of computing $Q_\Gamma(d)$, and then compute $Q_\delta(d)$ for $\delta = 1$ and 2.

Example 3.5. Let Γ be the second template of cogenus 2 in Figure 2. Note that Γ is the same long-edge graph G we considered in Examples 2.9 and 2.13. When d is sufficiently large, $Q_\Gamma(d) = Q^{d,\Gamma}$ is defined by (3.2). Hence, using the results for $\Phi(\Gamma_{(k)})$ given in Example 2.13 and the data in Figure 2, we get

$$Q_\Gamma(d) = 16 \cdot \left(0 + \left(-\frac{1}{2} \right) + (-2) + \sum_{k=4}^{d+0-1} -\frac{1}{2}(3k-5) \right) = -12d^2 + 52d - 56.$$

Example 3.6. Let $\delta = 1$. There are two templates of cogenus 1 as listed in Figure 2. We denote them by Γ_1 and Γ_2 in the order as listed in the Figure. We compute $Q_{\Gamma_1}(d)$ and $Q_{\Gamma_2}(d)$ similarly as shown in Example 3.5 but omit details of how we obtain $\Phi((\Gamma_i)_{(k)})$ for $k < k_{\min}(\Gamma_i)$.

$$\begin{aligned} Q_{\Gamma_1}(d) &= 4 \cdot \left(0 + \sum_{k=2}^{d+0-1} (k-1) \right) = 2d^2 - 6d + 4, \\ Q_{\Gamma_2}(d) &= 1 \cdot \left(\sum_{k=1}^{d+1-2} (2k+1) \right) = d^2 - 1. \end{aligned}$$

Therefore,

$$Q_1(d) = Q_{\Gamma_1}(d) + Q_{\Gamma_2}(d) = 3d^2 - 6d + 3 = 3(d-1)^2.$$

Example 3.7. Let $\delta = 2$. There are seven templates of cogenus 2 as listed in Figure 2. We denote them by $\Gamma_1, \Gamma_2, \dots, \Gamma_6$, and Γ_7 in the order as listed in the Figure. As in the previous example, we compute $Q_{\Gamma_i}(d)$ without details of how we obtain $\Phi((\Gamma_i)_{(k)})$ for $k < k_{\min}(\Gamma_i)$.

$$\begin{aligned} Q_{\Gamma_1}(d) &= 9 \cdot \left(0 + 0 + \sum_{k=3}^{d+0-1} (k-2) \right) = \frac{1}{2}(9d^2 - 45d + 54), \\ Q_{\Gamma_2}(d) &= 16 \cdot \left(0 + \left(-\frac{1}{2} \right) + (-2) + \sum_{k=4}^{d+0-1} -\frac{1}{2}(3k-5) \right) = -12d^2 + 52d - 56, \\ Q_{\Gamma_3}(d) &= 1 \cdot \left(-\frac{9}{2} + \sum_{k=2}^{d+1-2} -\frac{1}{2}(6k+1) \right) = -\frac{1}{2}(3d^2 - 2d + 1), \\ Q_{\Gamma_4}(d) &= 4 \cdot \left(0 + (-5) + \sum_{k=3}^{d+1-2} (-3k+1) \right) = -6d^2 + 10d + 4, \\ Q_{\Gamma_5}(d) &= 4 \cdot \left(-3 + \sum_{k=2}^{d+0-2} (-3k) \right) = -6d^2 + 18d - 12. \end{aligned}$$

Finally, by comparing formulas

$$Q_{\Gamma_6}(d) = 1 \cdot \left(\sum_{k=1}^{d+1-3} (3k+3) \right) \quad \text{and} \quad Q_{\Gamma_7}(d) = 1 \cdot \left(\sum_{k=1}^{d+1-3} (-3k-3) \right),$$

we see that $Q_{\Gamma_6}(d) + Q_{\Gamma_7}(d) = 0$ without calculation. Therefore,

$$Q_2(d) = Q_{\Gamma_1}(d) + Q_{\Gamma_2}(d) + Q_{\Gamma_3}(d) + Q_{\Gamma_4}(d) + Q_{\Gamma_5}(d) = -\frac{1}{2}(42d^2 - 117d + 75).$$

4. REFORMULATION OF THE RESULTS

In the previous section, we proved Corollary 1.3 using Corollary 2.17 and Theorem 2.12. Note that Corollary 2.17 is a consequence of Lemma 2.15. The goal of this section is to prove Lemma 2.15 and to finish the first reduction step in proving our main result Theorem 2.12.

There are two parts of this section. In the first part, we introduce τ -graphs, which generalize long-edge graphs. We then extend definitions and results of long-edge graphs to τ -graphs, and restate Lemma 2.15 and Theorem 2.12 in the setting of τ -graphs. The description of τ -graphs enables us to consider generating functions of functions Φ_d and Φ in certain forms, which will be used in the second part of this section to prove Lemma 2.15 and reduce Theorem 2.12 to a result on generating functions (Theorem 4.11).

We start with some basic combinatorial definitions and notation that will be used in the rest of the paper. $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers and $\mathbb{P} = \{1, 2, 3, \dots\}$ is the set of positive integers. Given a positive integer ℓ , we denote by $[\ell]$ the set $\{1, 2, \dots, \ell\}$.

We define $(\mathbb{N}^m)^* := \mathbb{N}^m \setminus 0$, the set of all tuples of m nonnegative integers except the tuple $(0, 0, \dots, 0)$.

For any $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m$, we define

$$\mathbf{x}^{\mathbf{n}} := x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}, \quad (-1)^{\mathbf{n}} = (-1)^{\sum_{i=1}^m n_i}.$$

Hence, we can write $(-\mathbf{x})^{\mathbf{n}}$ for $(-1)^{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$.

Suppose $\mathbf{n}_1 + \mathbf{n}_2 + \cdots + \mathbf{n}_i = \mathbf{n}$. We say $(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_i)$ is an i -composition of \mathbf{n} if $\mathbf{n}_1, \dots, \mathbf{n}_i \in (\mathbb{N}^m)^*$; we say $(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_i)$ is a *weak* i -composition of \mathbf{n} if $\mathbf{n}_1, \dots, \mathbf{n}_i \in \mathbb{N}^m$.

An alternative way of defining (long-edge) graphs. Each edge e of a (long-edge) graph G contains two pieces of information: its weight $\rho(e)$ and its adjacent vertices. For convenience in defining the statistics $\lambda_j(G)$, we use the set $I(e) = \{a+1, a+2, \dots, b\}$ to represent the edge $\{a, b\}$ with $a < b$. In this case, we say e is of *type* $(I(e), \rho(e))$. We will use this representation to describe edges of our graphs.

Definition 4.1. Fixing a positive integer m , let I_1, \dots, I_m be subsets of \mathbb{N} and $r_1, \dots, r_m \in \mathbb{P}$. For each $1 \leq i \leq m$, let $t_i = (I_i, r_i)$. We may assume t_1, \dots, t_m are distinct. Let $\tau = (t_1, \dots, t_m)$.

For any $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m$, we denote by $G_\tau(\mathbf{n})$ the graph on vertex set \mathbb{N} that has n_i edges of type t_i for each i . We call such a graph a τ -graph.

Given a τ -graph $G = G_\tau(\mathbf{n})$, we define its *multiplicity* to be

$$\mu_\tau(\mathbf{n}) := \prod_{i=1}^m ((r_i)^2)^{n_i},$$

and its *cogenu*s to be

$$\delta_\tau(\mathbf{n}) := \sum_{i=1}^m (r_i |I_i| - 1).$$

Remark 4.2. Note that if we require $\tau = (t_1, \dots, t_m)$ to satisfy that for each $1 \leq i \leq m$,

- (1) the set I_i is a set of consecutive integers, and
- (2) the product $r_i |I_i|$ is greater than 1,

we recover the definition of long-edge graphs. In particular, the definitions of multiplicity and cogenu agree with what we have defined before for long-edge graphs. Hence, τ -graphs generalize long-edge graphs.

Strictly speaking, without condition (1), a τ -graph is not a graph in the usual sense. However, most of the arguments appearing in this paper work without the restrictions (1) and/or (2).

In this paper, whenever we talk about long-edge graphs or templates, we will assume τ satisfies these two conditions without explicitly stating it.

Example 4.3. (1) Suppose $m = 1$ and $\tau = (t_1) = ((I, r))$, where $I = \{1\}$ and $r \in \mathbb{P}$. Then $G_\tau(n)$ is the graph with n edges of weight r connecting vertices 0 and 1.
(2) Suppose $m = 2$ and $\tau = (t_1, t_2) = ((I_1, r_1), (I_2, r_2))$, where $I_1 = \{1\}$, $I_2 = \{1, 2\}$ and $r_1, r_2 \in P$. Then $G_\tau(n_1, n_2)$ is the graph with n_1 edges of weight r_1 connecting 0 and 1 and n_2 edges of weight r_2 connecting vertices 0 and 2.

We can naturally extend all the definitions for long-edge graphs, such as allowability, templates and shifted graphs, to τ -graphs. For convenience, we write

$$\lambda_j(\tau, \mathbf{n}) := \lambda_j(G_\tau(\mathbf{n})) = \sum r_i n_i,$$

summing over all $i : 1 \leq i \leq m$ such that $j \in I_i$. Then we write

$$(4.1) \quad k_{\min}(\tau, \mathbf{n}) := k_{\min}(G_\tau(\mathbf{n})) = \max(0, \max\{\lambda_j(\tau, \mathbf{n}) - j + 1\}).$$

We have the following two lemmas corresponding to Lemmas 2.6 and 2.10.

Lemma 4.4. $G_\tau(\mathbf{n})_{(k)}$ is allowable if and only if $k \geq k_{\min}(\tau, \mathbf{n})$.

Lemma 4.5. For a fixed $\mathbf{n} \in \mathbb{N}^m$, for any $k \geq k_{\min}(\tau, \mathbf{n})$, (so $G_\tau(\mathbf{n})_{(k)}$ is allowable), the values $P(G_\tau(\mathbf{n})_{(k)})$ are given by a polynomial in k whose degree is $|\mathbf{n}| = n_1 + n_2 + \dots + n_m$, which is the number of edges in $G_\tau(\mathbf{n})$.

The proof for Lemma 2.6 can be easily extended to Lemma 4.4. Lemma 2.10 was proved in [6, Lemma 5.8]. Although Lemma 5.8 in [6] was only stated for templates, the proof works for any long-edge graph as well as any τ -graph. We will include a proof of Lemma 4.5 in the next section.

Finally, we rewrite the definitions of $\Phi_d(G)$ and $\Phi(G)$ given in (2.5) and (2.7), and then restate Lemma 2.15 and Theorem 2.12 in stronger versions.

Definition 4.6. Let $\mathbf{n} \in (\mathbb{N}^*)^m$. Define

$$(4.2) \quad \Phi_d(G_\tau(\mathbf{n})) := \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_i)} \prod_{j=1}^i P_d(G_\tau(\mathbf{n}_j)),$$

$$(4.3) \quad \Phi(G_\tau(\mathbf{n})) := \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_i)} \prod_{j=1}^i P(G_\tau(\mathbf{n}_j)).$$

Here for both equations, the second summation is over all the i -compositions of \mathbf{n} .

Note that $G_\tau(\mathbf{n})_{(k)}$ is not a τ -graph, but is a $(\tau + k)$ -graph where we consider

$$\tau + k = (t_1, \dots, t_m) + k = (t_1 + k, \dots, t_m + k), \text{ and } t_i + k = (\{a + k \mid a \in I_i\}, r_i).$$

Hence, $G_\tau(\mathbf{n})_{(k)} = G_{\tau+k}(\mathbf{n})$. Thus, it's easy to verify that

$$(4.4) \quad \Phi(G_\tau(\mathbf{n})_{(k)}) = \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_i)} \prod_{j=1}^i P(G_\tau(\mathbf{n}_j)_{(k)}).$$

Lemma 4.7. *Suppose $\tau = (t_1, \dots, t_m)$ where $t_i = (I_i, r_i)$ has the property that there exists $m' : 1 \leq m' < m$ such that for any $1 \leq i \leq m'$ and $m' + 1 \leq i' \leq m$, we have that I_i and $I_{i'}$ are disjoint. Then*

$$\Phi_d(G_\tau(\mathbf{n})) = 0, \quad \forall \mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2) \in (\mathbb{N}^{m'})^* \times (\mathbb{N}^{m-m'})^*.$$

Note that this lemma implies Lemma 2.15 because any long-edge graph that is not a shifted template can be described as $G_\tau(\mathbf{n})$ for some τ and $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2)$ as in the lemma.

We then restate Theorem 2.12 using the language of τ -graphs.

Theorem 4.8. *For any τ and any $\mathbf{n} \in (\mathbb{N}^m)^*$, the function $\Phi(G_\tau(\mathbf{n})_{(k)})$ is linear in k for $k \geq k_{\min}(\tau, \mathbf{n})$.*

Generating functions. We first state the following basic fact on generating functions: suppose $f(\mathbf{n}), g(\mathbf{n})$ are defined for $\mathbf{n} \in (\mathbb{N}^m)^*$. Then

$$\begin{aligned} g(\mathbf{n}) &= \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_i)} \prod_{j=1}^i f(\mathbf{n}_j), \quad \forall \mathbf{n} \\ &\iff \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} g(\mathbf{n}) \mathbf{x}^{\mathbf{n}} = \log \left(1 + \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} f(\mathbf{n}) \mathbf{x}^{\mathbf{n}} \right) \end{aligned}$$

Proof of Lemma 4.7. Fix τ and d . Recall that functions Φ_d and P_d satisfy (4.2). For convenience, we set $P_d(G_\tau(0)) = 1$. Hence,

$$\sum_{\mathbf{n} \in (\mathbb{N}^m)^*} \Phi_d(G_\tau(\mathbf{n})) \mathbf{x}^{\mathbf{n}} = \log \left(1 + \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} P_d(G_\tau(\mathbf{n})) \mathbf{x}^{\mathbf{n}} \right) = \log \left(\sum_{\mathbf{n} \in \mathbb{N}^m} P_d(G_\tau(\mathbf{n})) \mathbf{x}^{\mathbf{n}} \right).$$

However, by the assumption of τ , we see that there exist functions $f_1(\mathbf{n}_1)$ and $f_2(\mathbf{n}_2)$ such that for any $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2) \in (\mathbb{N}^{m'}) \times (\mathbb{N}^{m-m'})$, we have

$$P_d(G_\tau(\mathbf{n})) = f_1(\mathbf{n}_1) \cdot f_2(\mathbf{n}_2).$$

Let $\mathbf{x} = (y_1, \dots, y_{m'}, z_1, \dots, z_{m-m'})$. Then

$$\sum_{(\mathbf{n}_1, \mathbf{n}_2) \in (\mathbb{N}^m)^*} \Phi_d(G_\tau(\mathbf{n})) \mathbf{y}^{\mathbf{n}_1} \mathbf{z}^{\mathbf{n}_2} = \log \left(\sum_{(\mathbf{n}_1, \mathbf{n}_2) \in \mathbb{N}^m} f_1(\mathbf{n}_1) \cdot f_2(\mathbf{n}_2) \mathbf{y}^{\mathbf{n}_1} \mathbf{z}^{\mathbf{n}_2} \right)$$

$$= \log \left(\sum_{\mathbf{n}_1} f_1(\mathbf{n}_1) \mathbf{y}^{\mathbf{n}_1} \right) + \log \left(\sum_{\mathbf{n}_2} f_2(\mathbf{n}_2) \mathbf{z}^{\mathbf{n}_2} \right).$$

Since the last expression only involves terms $c_{\mathbf{n}_1, \mathbf{n}_2} \mathbf{y}^{\mathbf{n}_1} \mathbf{z}^{\mathbf{n}_2}$ with one of \mathbf{n}_1 and \mathbf{n}_2 being zero, the conclusion of the lemma follows. \square

Before discussing Theorem 4.8, we define two relevant polynomial functions.

Definition 4.9. Let $p_\tau(\mathbf{n}, k)$ be the polynomial in k that computes $P(G_\tau(\mathbf{n})_{(k)})$ for $k \geq k_{\min}(\tau, \mathbf{n})$. Since it is a polynomial, we can extend it to any $k \in \mathbb{Z}$.

We then define another polynomial in k :

$$(4.5) \quad \varphi_\tau(\mathbf{n}, k) := \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_i)} \prod_{j=1}^i p_\tau(\mathbf{n}_j, k),$$

where the second summation is over all the i -compositions of \mathbf{n} .

Corollary 4.10. For a fixed $\mathbf{n} \in \mathbb{N}^m$, for any $k \geq k_{\min}(\tau, \mathbf{n})$, the values $\Phi(G_\tau(\mathbf{n})_{(k)})$ are given by the polynomial $\varphi_\tau(\mathbf{n}, k)$.

Proof. Note that if $G_\tau(\mathbf{n})_{(k)}$ is allowable, for any i -composition $(\mathbf{n}_1, \dots, \mathbf{n}_i)$ of \mathbf{n} , we have that $G_\tau(\mathbf{n}_j)_{(k)}$ is allowable for any j . Hence, (4.4) becomes

$$(4.6) \quad \Phi(G_\tau(\mathbf{n}_j)_{(k)}) = \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \sum_{(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_i)} \prod_{j=1}^i p_\tau(\mathbf{n}_j, k).$$

Thus, the conclusion follows. \square

One sees that Theorem 4.8 just says that this polynomial $\varphi_\tau(\mathbf{n}, k)$ actually is linear in k .

Because of (4.5), it is natural to consider the following generating function

$$(4.7) \quad \mathcal{P}_{\tau, k}(\mathbf{x}) := 1 + \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} p_\tau(\mathbf{n}, k) \mathbf{x}^{\mathbf{n}}.$$

Then

$$(4.8) \quad \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} \varphi_\tau(\mathbf{n}, k) \mathbf{x}^{\mathbf{n}} = \log(\mathcal{P}_{\tau, k}(\mathbf{x})).$$

We have the following theorem for $\mathcal{P}_{\tau, k}(\mathbf{x})$.

Theorem 4.11. Let $\tau = (t_1, \dots, t_m)$, where $t_i = (I_i, m_i)$. Let ℓ be a positive integer such that each $I_i \subseteq [\ell]$. Then there exist formal power series $G_\tau(\mathbf{x})$ and $H_{\tau, \ell}(\mathbf{x})$ with $G_\tau(\mathbf{0}) = H_{\tau, \ell}(\mathbf{0}) = 1$ such that for any $k \in \mathbb{Z}$,

$$\mathcal{P}_{\tau, k}(\mathbf{x}) = (G_\tau(-\mathbf{x}))^{-k-\ell} H_{\tau, \ell}(-\mathbf{x}).$$

Assuming the above theorem, we can prove Theorem 4.8.

Proof of Theorem 4.8. Let $g(\mathbf{n})$ and $h(\mathbf{n})$ be the coefficients of $\mathbf{x}^{\mathbf{n}}$ in $\log G_\tau(\mathbf{x})$ and $\log H_{\tau, \ell}(\mathbf{x})$ respectively. In other words,

$$\log G_\tau(\mathbf{x}) = \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} g(\mathbf{n}) \mathbf{x}^{\mathbf{n}} \quad \text{and} \quad \log H_{\tau, \ell}(\mathbf{x}) = \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} h(\mathbf{n}) \mathbf{x}^{\mathbf{n}}.$$

Then

$$\log(\mathcal{P}_{\tau,k}(\mathbf{x})) = (-k - \ell) \log G_{\tau}(-\mathbf{x}) + \log H_{\tau,\ell}(-\mathbf{x}) = \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} [(-k - \ell)g(\mathbf{n}) + h(\mathbf{n})] (-\mathbf{x})^{\mathbf{n}}.$$

By (4.8), we have

$$(4.9) \quad \varphi_{\tau}(\mathbf{n}, k) = (-1)^{\mathbf{n}} (-g(\mathbf{n})k + (h(\mathbf{n}) - \ell g(\mathbf{n}))),$$

which is a linear function in k for fixing \mathbf{n} . Hence, the conclusion follows from Corollary 4.10. \square

Therefore, we reduce the problem of proving Theorems 2.12 and 4.8 to proving Theorem 4.11.

5. POLYNOMIALITY AND RECIPROCITY

In this section, we will introduce a new combinatorial object: (τ, \mathbf{n}) -words. A special family of these words, denoted by $S_{\tau}(\mathbf{n}, t; \ell)$, is counted by a polynomial function that has a reciprocity connection to the polynomial $p_{\tau}(\mathbf{n}, k)$. Using this connection, we reduce our problem (of proving Theorem 4.11) to proving a result on the generating function of $S_{\tau}(\mathbf{n}, t; \ell)$ (Proposition 5.9). Although Lemma 4.5 (the polynomiality of $P(G_{\tau}(\mathbf{n})_{(k)})$) was proved in [6], its proof is relevant to the proof of the reciprocity result, so we include both in this section.

Throughout the rest of the paper, we fix $\tau = (t_1, \dots, t_m)$, where $t_i = (I_i, r_i)$. Let $l(\tau)$ be the smallest positive integer ℓ satisfying each $I_i \subseteq [\ell]$, equivalently,

$$l(\tau) := l(G_{\tau}(\mathbf{n})), \forall \mathbf{n} \in (\mathbb{N}^*)^m.$$

Fix an integer $\ell \geq l(\tau)$.

For any $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m$, recall that

$$\lambda_j(\tau, \mathbf{n}) = \sum r_i n_i,$$

summing over all $i : 1 \leq i \leq m$ such that $j \in I_i$. We often omit the arguments τ and \mathbf{n} and only write λ_j if there's no confusion.

Before proving Lemma 4.5, we use an example to demonstrate the basic idea of the proof.

Example 5.1. Suppose $m = 2$ and $\tau = (t_1, t_2) = ((I_1, r_1), (I_2, r_2))$, where $I_1 = \{1\}$, $I_2 = \{1, 2\}$ and $r_1, r_2 \in \mathbb{P}$. Let $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$. Then

$$\lambda_1 = r_1 n_1 + r_2 n_2, \lambda_2 = r_2 n_2, \text{ and } \lambda_j = 0 \text{ for } j \geq 3.$$

Thus, $k_{\min}(\tau, \mathbf{n}) = \max(0, \lambda_1, \lambda_2 - 1) = \lambda_1$. Hence, for $k \geq \lambda_1$, we have that $G_{\tau}(\mathbf{n})_{(k)}$ is allowable. The graph $\text{ext}(G_{\tau}(\mathbf{n})_{(k)})$ has $k - \lambda_1$ new edges connecting vertices k and $k + 1$ and $k + 1 - \lambda_2$ new edges connecting vertices $k + 1$ and $k + 2$. Except the n_2 edges of type t_2 , all the other edges in $\text{ext}(G_{\tau}(\mathbf{n})_{(k)})$ have length 1 and thus their placement between vertices is determined. Hence, we can count the total number of extended orderings (up to equivalence) by considering how many edges of type t_2 are placed between vertices k and $k + 1$ and how many are placed between vertices $k + 1$ and $k + 2$. Therefore, we get the formula

$$p_{\tau}(\mathbf{n}, k) = P(G_{\tau}(\mathbf{n})_{(k)}) = \sum_{a_{1,2} + a_{2,2} = n_2} \binom{(k - \lambda_1) + n_1 + a_{1,2}}{k - \lambda_1, n_1, a_{1,2}} \binom{(k + 1 - \lambda_2) + a_{2,2}}{a_{2,2}}.$$

In the above formula $a_{j,2}$ represents the number of edges of type t_2 placed between vertices $k + j - 1$ and $k + j$.

We introduce a terminology for the data $(a_{i,j})$ used in the above example.

Definition 5.2. Let $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m$ and $\mathbf{c} = (c_1, \dots, c_\ell) \in \mathbb{N}^\ell$ satisfying $\sum_{i=1}^m n_i = \sum_{j=1}^\ell c_j$. We say an $m \times \ell$ matrix $A = (a_{i,j})$ is a *contingency table with margin (\mathbf{n}, \mathbf{c})* if all the entries of A are nonnegative, the i th row sum of A is n_i and the j th column sum of A is c_j , i.e., the following conditions are satisfied:

$$a_{i,j} \in \mathbb{N}, \quad \forall i, j; \quad \sum_{j=1}^\ell a_{i,j} = n_i, \quad \forall 1 \leq i \leq m; \quad \sum_{i=1}^m a_{i,j} = c_j, \quad \forall 1 \leq j \leq \ell.$$

Moreover, we say $A = (a_{i,j})$ is τ -compatible if $a_{i,j} = 0$ unless $j \in I_i$.

Proof of Lemma 4.5. Suppose $k \geq k_{\min}(\tau, \mathbf{n})$. Then we have that $G_\tau(\mathbf{n})_{(k)}$ is allowable.

There are two kinds of edges in $\text{ext}(G_\tau(\mathbf{n})_{(k)})$:

- (i) The original weighted edges in $G_\tau(\mathbf{n})_{(k)}$: for each $1 \leq i \leq m$, there are n_i edges of type $(I_i + k, r_i)$. (Here $I_i + k = \{j + k \mid j \in I_i\}$.)
- (ii) The new additional unweighted edges: For each $1 \leq j \leq \ell$, there are $(k + j - 1) - \lambda_j$ new unweighted edges connecting vertices $k + j - 1$ and $k + j$.

Given any extended ordering o of $G_\tau(\mathbf{n})_{(k)}$, if for any $1 \leq i \leq m$ and any $1 \leq j \leq \ell$, let $a_{i,j}$ be the number of edges of type t_i appearing between vertices $k + j - 1$ and $k + j$ in the ordering o , and let $c_j = \sum_{i=1}^m a_{i,j}$ be the number of all the weighted edges appearing between $k + j - 1$ and $k + j$, then the matrix $A = (a_{i,j})$ is a contingency table of margin (\mathbf{n}, \mathbf{c}) , where $\mathbf{c} = (c_1, \dots, c_\ell)$. We say A is the *contingency table corresponding to the ordering o* .

Naturally, we group the extended orderings by the contingency tables they correspond to. Thus, we can count the number of extended orderings (up to equivalence) by:

$$\begin{aligned} & P(G_\tau(\mathbf{n})_{(k)}) \\ &= \sum_{\mathbf{c}} \sum_A \# \text{extended orderings (up to equivalence) corresponding to the contingency table } A, \end{aligned}$$

where the first summation is over all the vectors $\mathbf{c} = (c_1, \dots, c_\ell) \in \mathbb{N}^\ell$ satisfying $\sum c_j = \sum n_i$, and the second summation is over all the contingency table A of margin (\mathbf{n}, \mathbf{c}) .

Fixing a contingency table $A = (a_{i,j})$ with margin (\mathbf{n}, \mathbf{c}) , we try to figure out how many ways are there to construct a corresponding extended ordering. For each j , the edges between the vertices $k + j - 1$ and $k + j$ include:

- $a_{i,j}$ edges of type t_i for each $i : 1 \leq i \leq m$.
- $k + j - 1 - \lambda_j$ unweighted edges.

Therefore, the number of ways to order the edges between the vertices $k + j - 1$ and $k + j$ is

$$\binom{k + j - 1 - \lambda_j + c_j}{c_j} \binom{c_j}{a_{1,j}, a_{2,j}, \dots, a_{m,j}}.$$

Hence, the number of extended orderings (up to equivalence) corresponding to the contingency table A is given by

$$\prod_{j=1}^{\ell} \binom{k+j-1-\lambda_j+c_j}{c_j} \binom{c_j}{a_{1,j}, a_{2,j}, \dots, a_{m,j}}.$$

Therefore,

$$(5.1) \quad P(G_{\tau}(\mathbf{n})_{(k)}) = \sum_{\mathbf{c}} \sum_A \prod_{j=1}^{\ell} \binom{k+j-1-\lambda_j+c_j}{c_j} \binom{c_j}{a_{1,j}, a_{2,j}, \dots, a_{m,j}}.$$

Clearly, this is a polynomial in k whose degree is $\sum_{j=1}^{\ell} c_j = \sum_{i=1}^m n_i$. \square

We now introduce (τ, \mathbf{n}) -words. Recall that we have fixed $\ell \geq l(\tau)$, so $I_i \subseteq [\ell]$ for each i .

Definition 5.3. A (τ, \mathbf{n}) -word is an ordered tuple of ℓ words (w_1, \dots, w_{ℓ}) satisfying the following conditions:

- a) Each w_j is a sequence of letters chosen from s_0, s_1, \dots, s_m where repetition is allowed.
- b) For each $1 \leq i \leq m$, the total number of s_i appearing in all the words is n_i .
- c) For each $1 \leq i \leq m$, the letter s_i can only occur in words w_j if $j \in I_i$.

Given $L_1, \dots, L_{\ell} \in \mathbb{N}$, we denote by $S_{\tau}(\mathbf{n}; L_1, \dots, L_{\ell})$ the set of all the (τ, \mathbf{n}) -words (w_1, \dots, w_{ℓ}) where the length of w_j is L_j .

We usually choose $\ell = l(\tau)$. However, it is not hard to see that there is a natural one-to-one correspondence between the $S_{\tau}(\mathbf{n}; L_1, \dots, L_{\ell})$ and the set $S_{\tau}(\mathbf{n} : L_1, \dots, L_{\ell}, L_{\ell+1}, \dots, L_{\ell'})$ for any $\ell' > \ell$, and $L_1, \dots, L_{\ell'} \in \mathbb{N}$, since for any $(w_1, \dots, w_{\ell}, w_{\ell+1}, \dots, w_{\ell'}) \in S_{\tau}(\mathbf{n} : L_1, \dots, L_{\ell}, L_{\ell+1}, \dots, L_{\ell'})$, the word w_j for $\ell < j \leq \ell'$ is just a sequence of letter s_0 's. Therefore, in some sense the choice of ℓ is not important for the general definition of (τ, \mathbf{n}) -words as long as $\ell \geq l(\tau)$.

Lemma 5.4. The cardinality of $S_{\tau}(\mathbf{n}; L_1, \dots, L_{\ell})$ is

$$\sum_{\mathbf{c}} \sum_A \prod_{j=1}^{\ell} \binom{L_j}{c_j} \binom{c_j}{a_{1,j}, a_{2,j}, \dots, a_{m,j}},$$

where the first summation is over all the vectors $\mathbf{c} = (c_1, \dots, c_{\ell}) \in \mathbb{N}^{\ell}$ satisfying $\sum c_j = \sum n_i$, and the second summation is over all the contingency tables A of margin (\mathbf{n}, \mathbf{c}) .

Proof. The idea of the proof is very similar to that of Lemma 4.5, thus is omitted. \square

Definition 5.5. Fixing τ , \mathbf{n} and ℓ , for any $t \in \mathbb{N}$, we denote by $S_{\tau}(\mathbf{n}, t; \ell)$ the set of all the (τ, \mathbf{n}) -words (w_1, \dots, w_{ℓ}) where the length of w_j is $t + \ell - j + \lambda_j$, i.e.,

$$S_{\tau}(\mathbf{n}, t; \ell) := S_{\tau}(\mathbf{n}; t + \ell - 1 + \lambda_1, t + \ell - 2 + \lambda_2, \dots, t + \ell - \ell + \lambda_{\ell}).$$

Note that the cardinality of $S_{\tau}(\mathbf{n}, t; \ell)$ depends on the choice of ℓ ; thus we include ℓ as an argument for the notation of the set.

Corollary 5.6. The cardinality of $S_{\tau}(\mathbf{n}, t; \ell)$ is

$$(5.2) \quad \sum_{\mathbf{c}} \sum_A \prod_{j=1}^{\ell} \binom{t + \ell - j + \lambda_j}{c_j} \binom{c_j}{a_{1,j}, a_{2,j}, \dots, a_{m,j}},$$

where the first summation is over all the vectors $\mathbf{c} = (c_1, \dots, c_\ell) \in \mathbb{N}^\ell$ satisfying $\sum c_j = \sum n_i$, and the second summation is over all the contingency table A of margin (\mathbf{n}, \mathbf{c}) .

Hence, $|S_\tau(\mathbf{n}, t; \ell)|$ is a polynomial in t .

Definition 5.7. We define $s_\tau(\mathbf{n}, t; \ell)$ to be the polynomial that computes $|S_\tau(\mathbf{n}, t; \ell)|$ when $t \in \mathbb{N}$. Since $s_\tau(\mathbf{n}, t; \ell)$ is a polynomial, we can extend it to $t \in \mathbb{Z}$.

We also define the generating function of $s_\tau(\mathbf{n}, t; \ell)$:

$$\mathcal{S}_{\tau, t; \ell}(\mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{N}^m} s_\tau(\mathbf{n}, t; \ell) \mathbf{x}^{\mathbf{n}} = 1 + \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} s_\tau(\mathbf{n}, t; \ell) \mathbf{x}^{\mathbf{n}}.$$

We now state the reciprocity formulas for $p_\tau(\mathbf{n}, k)$ and $s_\tau(\mathbf{n}, t; \ell)$ and their generating functions.

Lemma 5.8 (Reciprocity). *For any fixed \mathbf{n} ,*

$$(5.3) \quad p_\tau(\mathbf{n}, k) = (-1)^{\mathbf{n}} s_\tau(\mathbf{n}, -k - \ell; \ell).$$

Hence,

$$(5.4) \quad \mathcal{P}_{\tau, k}(\mathbf{x}) = \mathcal{S}_{\tau, -k - \ell; \ell}(-\mathbf{x}).$$

Proof. Note that $p_\tau(\mathbf{n}, k)$ is defined by (5.1) and $s_\tau(\mathbf{n}, t; \ell)$ is defined by (5.2). Hence, it is enough to show that

$$\prod_{j=1}^{\ell} \binom{k + j - 1 - \lambda_j + c_j}{c_j} = (-1)^{\mathbf{n}} \prod_{j=1}^{\ell} \binom{(-k - \ell) + \ell - j + \lambda_j}{c_j}.$$

Applying the reciprocity formula (cf. Formula (1.21) in [12])

$$\binom{-x}{n} = (-1)^n \binom{x + n - 1}{n}$$

we get

$$\binom{k + j - 1 - \lambda_j + c_j}{c_j} = (-1)^{c_j} \binom{-k - j + \lambda_j}{c_j}.$$

Hence,

$$\prod_{j=1}^{\ell} \binom{k + j - 1 - \lambda_j + c_j}{c_j} = \prod_{j=1}^{\ell} (-1)^{c_j} \binom{-k - j + \lambda_j}{c_j}.$$

However, $\sum_{j=1}^{\ell} c_j = \sum_{i=1}^m n_i$. Equation (5.3) follows. We use (5.3) to prove (5.4):

$$\begin{aligned} \mathcal{P}_{\tau, k}(\mathbf{x}) &= \sum_{\mathbf{n}} p_\tau(\mathbf{n}, k) \mathbf{x}^{\mathbf{n}} = \sum_{\mathbf{n}} (-1)^{\mathbf{n}} s_\tau(\mathbf{n}, -k - \ell; \ell) \mathbf{x}^{\mathbf{n}} \\ &= \sum_{\mathbf{n}} s_\tau(\mathbf{n}, -k - \ell; \ell) (-\mathbf{x})^{\mathbf{n}} = \mathcal{S}_{\tau, -k - \ell; \ell}(-\mathbf{x}) \end{aligned}$$

□

By (5.4), one sees that Theorem 4.11 is equivalent to the following proposition:

Proposition 5.9. *There exist formal power series $G_\tau(\mathbf{x})$ and $H_{\tau;\ell}(\mathbf{x})$ with $G_\tau(\mathbf{0}) = H_{\tau;\ell}(\mathbf{0}) = 1$ such that for any $t \in \mathbb{Z}$,*

$$\mathcal{S}_{\tau,t;\ell}(\mathbf{x}) = (G_\tau(\mathbf{x}))^t H_{\tau;\ell}(\mathbf{x}).$$

Note that the functions $G_\tau(\mathbf{x})$ and $H_{\tau;\ell}(\mathbf{x})$ in Proposition 5.9 and Theorem 4.11 are the same.

6. INGREDIENTS FOR PROVING PROPOSITION 5.9

In this section, we will present the two key ingredients for proving Proposition 5.9 and reduce the problem further to proving Lemma 6.7, which will be proved in the next section. First, we see that we only need to prove a weaker version of Proposition 5.9.

Proposition 6.1. *There exist formal power series $G_\tau(\mathbf{x})$ and $H_{\tau;\ell}(\mathbf{x})$ with $G_\tau(\mathbf{0}) = H_{\tau;\ell}(\mathbf{0}) = 1$ such that for any $t \in \mathbb{N}$, we have*

$$\mathcal{S}_{\tau,t;\ell}(\mathbf{x}) = (G_\tau(\mathbf{x}))^t H_{\tau;\ell}(\mathbf{x}).$$

Proof of that Proposition 6.1 implies Proposition 5.9. Given formal power series $G_\tau(\mathbf{x})$ and $H_{\tau;\ell}(\mathbf{x})$ with $G_\tau(\mathbf{0}) = H_{\tau;\ell}(\mathbf{0}) = 1$, it is easy to show that for any t ,

$$(G_\tau(\mathbf{x}))^t H_{\tau;\ell}(\mathbf{x}) = 1 + \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} f(\mathbf{n}, t) \mathbf{x}^{\mathbf{n}},$$

where $f(\mathbf{n}, t)$ is a polynomial in t for any fixed \mathbf{n} .

Assuming Proposition 6.1, we know that $f(\mathbf{n}, t)$ and $s_\tau(\mathbf{n}, t; \ell)$ are both polynomials and agree at infinitely many values of t 's. Therefore, they have to be the same polynomial. Hence, Proposition 5.9 follows. \square

Next we will see that by basic properties of ordinary generating functions, Proposition 6.1 is equivalent to the following lemma.

Lemma 6.2. *There exists functions $a(\mathbf{n})$ and $b(\mathbf{n})$ such that for any $t \in \mathbb{N}$,*

$$(6.1) \quad s_\tau(\mathbf{n}, t; \ell) = |S_\tau(\mathbf{n}, t; \ell)| = \sum_{(\mathbf{n}_0, \mathbf{n}_1, \dots, \mathbf{n}_t)} a(\mathbf{n}_1) a(\mathbf{n}_2) \cdots a(\mathbf{n}_t) b(\mathbf{n}_0),$$

where the summation is over all the weak $(t+1)$ -compositions of \mathbf{n} .

Moreover, $a(\mathbf{n})$ and $b(\mathbf{n})$ are positive for $\mathbf{n} \in \mathbb{N}^m$.

Now our problem becomes to finding two statistics related to (τ, \mathbf{n}) -words which provide the functions $a(\mathbf{n})$ and $b(\mathbf{n})$ we need in Equation (6.1). By setting $q = 0$ in (6.1), one notices that $b(\mathbf{n})$ should be the cardinality of $S_\tau(\mathbf{n}, 0; \ell)$. We devote the rest of section to defining the object that is enumerated by the function $a(\mathbf{n})$ we need in (6.1).

Definition 6.3. Given a (τ, \mathbf{n}) -word $\mathbf{w} = (w_1, \dots, w_\ell)$, we associate a *height function* with it:

$$h(\mathbf{w}) = (h_1, \dots, h_\ell) = (h_1(\mathbf{w}), \dots, h_\ell(\mathbf{w})),$$

where $h_j = h_j(\mathbf{w})$ is defined by

$$h_j = h_j(\mathbf{w}) = (-1) \cdot \#(\text{letter } s_0\text{'s in } w_j) + \sum_{i: j \in I_i} (r_i - 1) \cdot \#(\text{letter } s_i\text{'s in } w_j)$$

$$+ \sum_{i: j \in I_i} r_i \cdot \#(\text{letter } s_i \text{'s not in } w_j).$$

Another way to look at the height function is that each s_0 appearing in w_j contributes -1 to the height number h_j , and for each $i : j \in I_i$ any letter s_i appearing in w_j contributes $(r_i - 1)$ to h_j and any letter s_i appearing in words other than w_j contributes r_i to h_j . (Note that if s_i appears in w_j , we must have that $j \in I_i$.)

Definition 6.4. Let $\mathbf{u} = (u_1, \dots, u_\ell)$ and $\mathbf{v} = (v_1, \dots, v_\ell)$ be a (τ, \mathbf{n}_1) -word and a (τ, \mathbf{n}_2) -word respectively. The *concatenation* of \mathbf{u} and \mathbf{v} is defined to be

$$\mathbf{u} \circ \mathbf{v} = (u_1 v_1, \dots, u_\ell v_\ell),$$

which is clearly a $(\tau, \mathbf{n}_1 + \mathbf{n}_2)$ -word.

Suppose $\mathbf{w} = \mathbf{u} \circ \mathbf{v}$. We say \mathbf{u} is an *initial subword* of \mathbf{w} .

Definition 6.5. A (τ, \mathbf{n}) -word $\mathbf{w} = (w_1, \dots, w_\ell)$ is *balanced* if $h(\mathbf{w}) = 0$.

A (τ, \mathbf{n}) -word $\mathbf{w} = (w_1, \dots, w_\ell)$ is *quasi-balanced* if $h(\mathbf{w}) = (-1, -1, \dots, -1)$.

We say a (τ, \mathbf{n}) -word \mathbf{w} has *negative height* if $h_j(\mathbf{w}) < 0$ for each j .

Note that quasi-balanced words are those words that have the biggest negative height.

Definition 6.6. We say a quasi-balanced (τ, \mathbf{n}) -word $\mathbf{w} = (w_1, \dots, w_\ell)$ is *irreducible* if \mathbf{w} does not have a proper initial subword that is also quasi-balanced.

We denote by $\text{iqb}(\tau, \mathbf{n})$ the set of all (τ, \mathbf{n}) -words that are irreducible quasi-balanced.

Although it seems to be natural to define irreducible balanced words in the same fashion, it turns out that not having a proper initial balanced subword is not the correct definition. The definition of irreducible balanced words will be given in next section.

Now we can state the result that leads to Lemma 6.2

Lemma 6.7. Let $t \in \mathbb{N}$. Suppose $\mathbf{w} = (w_1, \dots, w_\ell) \in S_\tau(\mathbf{n}, t; \ell)$. Then there is a unique way to decompose \mathbf{w} as

$$\mathbf{w} = \mathbf{u}_1 \circ \mathbf{u}_2 \circ \dots \circ \mathbf{u}_t \circ \mathbf{v},$$

such that $\mathbf{u}_1, \dots, \mathbf{u}_t$ are all irreducible quasi-balanced.

Moreover, given the above decomposition of \mathbf{w} , we have

$$h_j(\mathbf{v}) = -\ell + j, \quad \forall 1 \leq j \leq \ell.$$

Further, for $1 \leq i \leq t$, let $\mathbf{n}_i \in \mathbb{N}^m$ be such that \mathbf{u}_i is a (τ, \mathbf{n}_i) -word, and let $\mathbf{n}_0 = \mathbf{n} - \mathbf{n}_1 - \dots - \mathbf{n}_t$. Then \mathbf{v} is in $S_\tau(\mathbf{n}_0, 0; \ell)$.

Therefore, the decomposition induces a bijection

$$S_\tau(\mathbf{n}, t; \ell) \rightarrow \bigsqcup_{(\mathbf{n}_0, \mathbf{n}_1, \dots, \mathbf{n}_t)} \text{iqb}(\tau, \mathbf{n}_1) \times \dots \times \text{iqb}(\tau, \mathbf{n}_t) \times S_\tau(\mathbf{n}_0, 0; \ell),$$

where the disjoint union is over all the weak $(t+1)$ -compositions of \mathbf{n} .

The bijection described in the above lemma implies Lemma 6.2 and Proposition 6.1.

Proof of Lemma 6.2 and Proposition 6.1. Let

$$a(\mathbf{n}) = |\text{iqb}(\tau, \mathbf{n})| \text{ and } b(\mathbf{n}) = |S_\tau(\mathbf{n}, 0; \ell)|.$$

Then (6.1) follows from Lemma 6.7. It is not hard to see that both $\text{iqb}(\tau, \mathbf{n})$ and $S_\tau(\mathbf{n}, 0; \ell)$ are nonempty sets. Hence, Lemma 6.2 follows.

Let

$$G_\tau(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}^m} a(\mathbf{n}) \mathbf{x}^{\mathbf{n}} \text{ and } H_{\tau, \ell}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}^m} b(\mathbf{n}) \mathbf{x}^{\mathbf{n}}.$$

As in the discussion following Definition 5.3, one sees that the cardinality of $\text{iqb}(\tau, \mathbf{n})$ does not depend on the choice of ℓ as long as $\ell \geq l(\tau)$. Hence, the function $G_\tau(\mathbf{x})$ is independent from ℓ .

It is clear that both $\text{iqb}(\tau, \mathbf{0})$ and $S_\tau(\mathbf{0}, 0; \ell)$ contain one element. Hence, $G_\tau(\mathbf{0}) = a(\mathbf{0}) = 1$ and $H_{\tau, \ell}(\mathbf{0}) = b(\mathbf{0}) = 1$. Thus, Proposition 6.1 follows from (6.1). \square

Hence, we only need to prove Lemma 6.7 to complete the proof of Proposition 5.9. In order to prove Lemma 6.7, we need to discuss more properties of the height functions and describe a general way to find initial subwords that are irreducible and quasi-balanced. We will discuss all of these topics in the next section.

7. DECOMPOSITION OF (τ, \mathbf{n}) -WORDS

The main result of this section is the following theorem.

Theorem 7.1. *Suppose $\mathbf{w} = (w_1, \dots, w_\ell)$ is a (τ, \mathbf{n}) -word with negative height. Then \mathbf{w} has a unique initial subword \mathbf{u} that is irreducible quasi-balanced.*

We first show that Lemma 6.7 follows from Theorem 7.1. We begin by giving an explicit description for the height function of words in $S_\tau(\mathbf{n}; L_1, \dots, L_\ell)$.

Lemma 7.2. *Let $\mathbf{w} = (w_1, \dots, w_\ell) \in S_\tau(\mathbf{n}; L_1, \dots, L_\ell)$. Then*

$$h_j(\mathbf{w}) = \lambda_j(\tau, \mathbf{n}) - L_j, \quad \forall 1 \leq j \leq \ell.$$

Proof. Let $a_{i,j}$ be the number of letters s_i appearing in the word w_j . Thus, there are $L_j - \sum_i a_{i,j}$ of s_0 appearing in w_j and $n_i - a_{i,j}$ of s_i not appearing in w_j . Then

$$\begin{aligned} h_j(\mathbf{w}) &= (-1) \cdot (L_j - \sum_i a_{i,j}) + \sum_i (r_i - 1) \cdot a_{i,j} + \sum_i r_i \cdot (n_i - a_{i,j}) \\ &= -L_j + \sum_i a_{i,j} + \sum_i r_i a_{i,j} - \sum_i a_{i,j} + \sum_i r_i n_i - \sum_i r_i a_{i,j} \\ &= \lambda_j(\tau, \mathbf{n}) - L_j, \end{aligned}$$

where all the summations in the above equation are over all i such that $j \in I_i$. \square

Corollary 7.3. *Suppose $\mathbf{w} = \mathbf{u} \circ \mathbf{v}$. Then*

$$h(\mathbf{w}) = h(\mathbf{u}) + h(\mathbf{v}).$$

Proof. This follows from Lemma 7.2 and the fact that

$$\lambda_j(\tau, \mathbf{n}_1 + \mathbf{n}_2) = \lambda_j(\tau, \mathbf{n}_1) + \lambda_j(\tau, \mathbf{n}_2).$$

(It is also possible to prove the corollary directly using the definition of the height function.) \square

The following Corollary is an immediate consequence of Lemma 7.2.

Corollary 7.4. Suppose $\mathbf{w} = (w_1, \dots, w_\ell)$ is a (τ, \mathbf{n}) -word and let $t \in \mathbb{N}$. Then $\mathbf{w} \in S_\tau(\mathbf{n}, t; \ell)$ if and only if

$$h_j(\mathbf{w}) = -t - \ell + j, \quad \forall 1 \leq j \leq \ell.$$

By Corollary 7.3, we have an immediate corollary to Theorem 7.1.

Corollary 7.5. Suppose $\mathbf{w} = (w_1, \dots, w_\ell)$ is a (τ, \mathbf{n}) -word and for some $t \in \mathbb{N}$, $h_j(\mathbf{w}) \leq -t$ for each j . Then there is a unique way to decompose \mathbf{w} as

$$\mathbf{w} = \mathbf{u}_1 \circ \mathbf{u}_2 \circ \dots \circ \mathbf{u}_t \circ \mathbf{v},$$

such that $\mathbf{u}_1, \dots, \mathbf{u}_t$ are all irreducible quasi-balanced.

Now if we assume Theorem 7.1, which implies Corollary 7.5, we are able to prove Lemma 6.7.

Proof of Lemma 6.7. Let $t \in \mathbb{N}$ and $\mathbf{w} \in S_\tau(\mathbf{n}, t; \ell)$. By Corollary 7.4, $h_j(\mathbf{w}) \leq -t$ for each $1 \leq j \leq \ell$. Therefore, we can apply Corollary 7.5 and conclude that there exists a unique decomposition of \mathbf{w} :

$$\mathbf{w} = \mathbf{u}_1 \circ \mathbf{u}_2 \circ \dots \circ \mathbf{u}_t \circ \mathbf{v},$$

such that $\mathbf{u}_1, \dots, \mathbf{u}_t$ are all irreducible quasi-balanced. Further, by Corollary 7.3, we have

$$h(\mathbf{v}) = h(\mathbf{w}) - h(\mathbf{u}_1) - \dots - h(\mathbf{u}_t) = h(\mathbf{w}) - (-1, \dots, -1) \times t.$$

Hence,

$$h_j(\mathbf{v}) = -\ell + j, \quad \forall 1 \leq j \leq \ell.$$

Therefore, by Corollary 7.4, $\mathbf{v} \in S_\tau(\mathbf{n}_0, 0; \ell)$. □

We will prove Theorem 7.1 in the rest of the section. The main idea is to describe an algorithm that finds an initial subword \mathbf{u} that is quasi-balanced and then show \mathbf{u} is irreducible.

Algorithm: Find-Irreducible-Quasi-Balanced-Subword (FIQBS)

Input: $\mathbf{w} = (w_1, \dots, w_\ell)$ a (τ, \mathbf{n}) -word with negative height, i.e., $h_j(\mathbf{w}) < 0$ for each $1 \leq j \leq \ell$.

- (1) Let $\mathbf{u} = (u_1, \dots, u_\ell) = (\emptyset, \dots, \emptyset)$ be the empty word and let $\mathbf{v} = (v_1, \dots, v_\ell) = \mathbf{w}$. (Clearly $h(\mathbf{u}) = (0, 0, \dots, 0)$.)
- (2) While $h(\mathbf{u}) \neq (-1, -1, \dots, -1)$:
 - (a) Pick one of the nonnegative entry of $h(\mathbf{u})$, say $h_j(\mathbf{u}) \geq 0$.
 - (b) Suppose the first letter in v_j is α . Remove α from v_j and add α to the end of u_j .
 - (c) Go back to (2).
- (3) Output (\mathbf{u}, \mathbf{v}) .

It is clear that if the algorithm works, at each step the pair (\mathbf{u}, \mathbf{v}) always has the property that $\mathbf{u} \circ \mathbf{v} = \mathbf{w}$, and when the algorithm terminates, the word \mathbf{u} satisfies that $h(\mathbf{u}) = (-1, -1, \dots, -1)$, so is quasi-balanced.

We first show that the algorithm works.

Lemma 7.6. At each step of the algorithm FIQBS, we always have that

$$h_j(\mathbf{u}) \in \mathbb{Z}_{\geq -1}, \quad \forall 1 \leq j \leq \ell.$$

Note that this lemma shows that whenever $h(\mathbf{u}) \neq (-1, \dots, -1)$, there always exists j such that $h_j(\mathbf{u}) \geq 0$. Thus, the algorithm won't run into trouble at step (2)/(a).

Proof. We prove this by induction. Initially, $h(\mathbf{u}) = (0, \dots, 0) \in (\mathbb{Z}_{\geq -1})^\ell$.

Suppose at the beginning of a loop inside (2), we have $h(\mathbf{u}) \in (\mathbb{Z}_{\geq -1})^\ell$. Since $h(\mathbf{u}) \neq (-1, \dots, -1)$, we can find a j such that $h_j(\mathbf{u}) \geq 0$. Suppose we run step (b) without problem and let α be the letter involved. To avoid confusion, we use \mathbf{u}' to denote the new \mathbf{u} we obtained in step (b). We want to show that $h(\mathbf{u}') \in (\mathbb{Z}_{\geq -1})^\ell$. There are two situations.

- If $\alpha = s_0$, then

$$h_{j'}(\mathbf{u}') = \begin{cases} h_{j'}(\mathbf{u}), & j' \neq j \\ h_{j'}(\mathbf{u}) - 1, & j' = j \end{cases}$$

- If $\alpha = s_i$ for some $1 \leq i \leq m$, then

$$h_{j'}(\mathbf{u}') = \begin{cases} h_{j'}(\mathbf{u}), & j' \notin I_i \\ h_{j'}(\mathbf{u}) + r_i, & j' = j \text{ (so } j' \in I_i) \\ h_{j'}(\mathbf{u}) + r_i - 1, & j' \in I_i, j' \neq j \end{cases}$$

In both cases, one checks that $h(\mathbf{u}) \in (\mathbb{Z}_{\geq -1})^\ell$ implies that $h(\mathbf{u}') \in (\mathbb{Z}_{\geq -1})^\ell$. \square

Lemma 7.7. *At each step of the algorithm FIQBS, if $h_j(\mathbf{u}) \geq 0$, then $u_j \neq w_j$.*

This lemma indicates that whenever $h_j(\mathbf{u}) \geq 0$, we must have that u_j is not the whole word w_j yet and thus v_j is not empty and we can pick the first letter of v_j . Hence, the algorithm won't run into trouble at step (2)/(b). We prove Lemma 7.7 using the following lemma.

Lemma 7.8. *Suppose \mathbf{u} is an initial subword of \mathbf{w} . For any $j : 1 \leq j \leq \ell$, if $u_j = w_j$, i.e., u_j is the whole word w_j , then $h_j(\mathbf{u}) \leq h_j(\mathbf{w})$.*

Proof. By the definition of the height function, if $u_j = w_j$, we have that

$$h_j(\mathbf{w}) - h_j(\mathbf{u}) = \sum_{i: j \in I_i} r_i \cdot (\#(\text{letters } s_i \text{'s in } \mathbf{w}) - \#(\text{letters } s_i \text{'s in } \mathbf{u})) \geq 0.$$

\square

Proof of Lemma 7.7. At each step of the algorithm, \mathbf{u} is always an initial subword of \mathbf{w} . Since the input \mathbf{w} has negative weight, we have $h_j(\mathbf{w}) < 0 \leq h_j(\mathbf{u})$. Hence, by Lemma 7.8, we conclude that $u_j \neq w_j$. \square

Lemma 7.9. *The algorithm FIQBS always terminates.*

Proof. Since the number of letters in \mathbf{u} increases by one each time we run the loop (a)-(c) inside step (2), and \mathbf{u} is an initial subword of \mathbf{w} , which has finitely many letters, the algorithm has to terminate at some point. \square

Lemmas 7.6, 7.7 and 7.9 show that our algorithm FIQBS is a well-defined algorithm. We finally discuss the properties of the output of the algorithm.

Lemma 7.10. *Suppose (\mathbf{u}, \mathbf{v}) is the output of the algorithm FIQBS taking input \mathbf{w} . The followings are true.*

- (i) *For any initial subword \mathbf{u}' of \mathbf{w} that is quasi-balanced, we must have that \mathbf{u} is an initial subword of \mathbf{u}' . Therefore, \mathbf{u} is irreducible quasi-balanced.*

- (ii) \mathbf{u} is obtained from a balanced (τ, \mathbf{n}) -word \mathbf{u}' by appending one s_0 to the end of each of ℓ words in \mathbf{u}' . In other words,

$$\mathbf{u} = \mathbf{u}' \circ (s_0, \dots, s_0),$$

where \mathbf{u}' is balanced.

Proof. (i) Assume to the contrary that \mathbf{u} is not an initial subword of \mathbf{u}' . For convenience, we name the list of \mathbf{u} 's created by the algorithm $\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, \dots, \mathbf{u}^{(k)}$. So we have $\mathbf{u}^{(0)}$ is the empty word and $\mathbf{u}^{(k)} = \mathbf{u}$.

Since $\mathbf{u}^{(0)}$, the empty word, is an initial subword of \mathbf{u}' by definition, there exists $i : 0 \leq i \leq k-1$ such that $\mathbf{u}^{(i)}$ is an initial subword of \mathbf{u}' and $\mathbf{u}^{(i+1)}$ is not an initial subword of \mathbf{u}' , where we have obtained $\mathbf{u}^{(i+1)}$ from $\mathbf{u}^{(i)}$ by running the while loop (2) in the algorithm once. Suppose during this loop, we take the nonnegative entry $h_j(\mathbf{u}^{(i)}) \geq 0$, and append a letter α to $u_j^{(i)}$. One sees that we must have that $u_j^{(i)} = u'_j$. Because \mathbf{u}' is quasi-balanced, it has negative height. Then by Lemma 7.8,

$$h_j(\mathbf{u}^{(i)}) \leq h_j(\mathbf{u}').$$

However, $h_j(\mathbf{u}^{(i)}) \geq 0$ and $h_j(\mathbf{u}') = -1$. This is a contradiction.

- (ii) It is enough to show that \mathbf{u} ends with (s_0, \dots, s_0) . However, we see that $h_j(\mathbf{u})$ decreases only when adding s_0 to u_j and increases when adding any other letter to u_j . Therefore, in order to have $h_j(\mathbf{u})$ become -1 , the last letter the algorithm adds to u_j has to be s_0 . □

Proof of Theorem 7.1. Suppose (\mathbf{u}, \mathbf{v}) is the output of the algorithm FIQBS taking input \mathbf{w} . By Lemma 7.10/(i), the word \mathbf{u} is an irreducible quasi-balanced initial subword of \mathbf{w} . Suppose \mathbf{u}' is also an irreducible quasi-balanced initial subword of \mathbf{w} . Then by Lemma 7.10/(i), we have \mathbf{u} is an initial subword of \mathbf{u}' . However, since \mathbf{u}' is irreducible, $\mathbf{u} = \mathbf{u}'$. Thus, the uniqueness follows. □

There are several consequences of Lemma 7.10/(ii).

Corollary 7.11. *Suppose \mathbf{w} is irreducible quasi-balanced. Then \mathbf{w} can be written as*

$$\mathbf{w} = \mathbf{u} \circ (s_0, \dots, s_0),$$

for some balanced (τ, \mathbf{n}) -word \mathbf{u} .

Proof. Since \mathbf{w} has negative height, we can run FIQBS with \mathbf{w} . The output must be $(\mathbf{w}, (\emptyset, \dots, \emptyset))$. Then the conclusion follows from Lemma 7.10/(ii). □

We can also give an alternative definition for irreducible quasi-balanced words.

Corollary 7.12. *Let \mathbf{w} be a quasi-balanced (τ, \mathbf{n}) -word. Then \mathbf{w} is irreducible if and only if \mathbf{w} does not have any proper initial subword with negative height.*

Proof. The direction “if” follows from the fact that any quasi-balanced word has negative height.

We prove the contrapositive of the direction “only if”. Suppose \mathbf{w} has a proper initial subword \mathbf{u} with negative height. Then we can run FIQBS on \mathbf{u} which finds an irreducible quasi-balanced word that is an initial subword of \mathbf{u} , hence is a proper initial subword of \mathbf{w} . Thus, \mathbf{w} is not irreducible quasi-balanced. □

Using this alternative definition, we define irreducible balanced words.

Definition 7.13. We say a balanced (τ, \mathbf{n}) -word \mathbf{w} is *irreducible* if \mathbf{w} does not have any proper initial subword with negative height.

We denote by $\text{ib}(\tau, \mathbf{n})$ the set of all (τ, \mathbf{n}) -words that are irreducible balanced.

We see that irreducible balanced words are exactly the balanced words that appeared in Corollary 7.11. Hence, we have the following result.

Lemma 7.14. *The map $\mathbf{w} \mapsto \mathbf{w} \circ (s_0, \dots, s_0)$ gives a bijection from $\text{ib}(\tau, \mathbf{n})$ to $\text{iqb}(\tau, \mathbf{n})$. Hence,*

$$|\text{ib}(\tau, \mathbf{n})| = |\text{iqb}(\tau, \mathbf{n})|.$$

Therefore, we can restate Lemma 6.7 in terms of irreducible balanced words.

Lemma 7.15. *Let $t \in \mathbb{N}$. Denote by $\mathbf{s}_0 = (s_0, \dots, s_0)$ the trivial irreducible quasi-balanced word. Suppose $\mathbf{w} = (w_1, \dots, w_\ell) \in S_\tau(\mathbf{n}, t; \ell)$. Then there is a unique way to decompose \mathbf{w} as*

$$\mathbf{w} = \mathbf{u}_1 \circ \mathbf{s}_0 \circ \mathbf{u}_2 \circ \mathbf{s}_0 \circ \dots \circ \mathbf{u}_t \circ \mathbf{s}_0 \circ \mathbf{v},$$

such that $\mathbf{u}_1, \dots, \mathbf{u}_t$ are all irreducible balanced words and \mathbf{v} is in $S_\tau(\mathbf{n}_0, 0; \ell)$.

Therefore, the decomposition induces a bijection

$$S_\tau(\mathbf{n}, t; \ell) \rightarrow \bigsqcup_{(\mathbf{n}_0, \mathbf{n}_1, \dots, \mathbf{n}_t)} \text{ib}(\tau, \mathbf{n}_1) \times \dots \times \text{ib}(\tau, \mathbf{n}_t) \times S_\tau(\mathbf{n}_0, 0; \ell),$$

where the disjoint union is over all the weak $(t+1)$ -compositions of \mathbf{n} .

8. EXAMPLES OF $\varphi(\mathbf{n}, k)$

Recall that $\varphi_\tau(\mathbf{n}, k)$ is the linear function described in Theorems 1.2 and 2.12 if $G = G_\tau(\mathbf{n})$. In this section, we first consider a family of simple examples for which we are able to describe $p_\tau(\mathbf{n}, k)$, $|\text{iqb}(\tau, \mathbf{n})|$ and $|S_\tau(\mathbf{n}, 0, \ell)|$ explicitly, and demonstrate the idea of how one might use this information to figure out an expression for $\varphi_\tau(\mathbf{n}, k)$ on a special subfamily of the examples. We then extend the result and give an expression for $\varphi_\tau(\mathbf{n}, k)$ for any τ that consists of one type of edges.

We start by considering the situation when $l(\tau) = 1$. Then every $I_i = \{1\}$. Suppose $\tau = (t_1, t_2, \dots, t_m)$, where $t_i = (\{1\}, r_i)$ and r_1, \dots, r_m are distinct positive integers. Then $G_\tau(\mathbf{n})$ has $\sum_i n_i$ edges, n_i of which have weight r_i . We have $\lambda_1(G_\tau(\mathbf{n})) = \sum_i r_i n_i$ and $\lambda_j(G_\tau(\mathbf{n})) = 0$ for all $j \geq 2$. Thus, $k_{\min}(\tau, \mathbf{n}) = \sum_i r_i n_i$.

For $k \geq \sum_i r_i n_i$, the set of edges connecting vertices k and $k+1$ in the graph $\text{ext}(G_\tau(\mathbf{n})_{(k)})$ consists of n_i edges of weight r_i for each i and $k - \sum_i r_i n_i$ unweighted edges. Therefore,

$$p_\tau(\mathbf{n}, k) = P(G_\tau(\mathbf{n})_{(k)}) = \binom{k - \sum_i r_i n_i + \sum_i n_i}{\sum_i n_i} \binom{\sum_i n_i}{n_1, n_2, \dots, n_m}.$$

Thus,

$$\mathcal{P}_{\tau, k}(\mathbf{x}) = 1 + \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} \binom{k - \sum_i r_i n_i + \sum_i n_i}{\sum_i n_i} \binom{\sum_i n_i}{n_1, n_2, \dots, n_m} \mathbf{x}^{\mathbf{n}},$$

and

$$\sum_{\mathbf{n} \in (\mathbb{N}^m)^*} \varphi_\tau(\mathbf{n}, k) \mathbf{x}^{\mathbf{n}} = \log \left(1 + \sum_{\mathbf{n} \in (\mathbb{N}^m)^*} \binom{k - \sum_i r_i n_i + \sum_i n_i}{\sum_i n_i} \binom{\sum_i n_i}{n_1, n_2, \dots, n_m} \mathbf{x}^{\mathbf{n}} \right).$$

Let $\ell = l(\tau) = 1$. In order to find functions $G_\tau(x)$ and $H_{\tau, \ell}(x)$ in Theorem 4.11, Proposition 5.9 and Proposition 6.1, we consider the corresponding (τ, \mathbf{n}) -word.

For the given setup, the (τ, \mathbf{n}) -words are actually the Lukasiewicz words in the literature. See Section 5.3 of [11]. It is known that there is a natural one-to-one correspondence between words \mathbf{w} in $\text{iqb}(\tau, \mathbf{n})$ and plane trees which have n_i internal vertices of degree r_i for each i and $1 + \sum_i (r_i - 1)n_i$ leaves. Using this, one obtains the cardinality for $\text{iqb}(\tau, \mathbf{n})$ (Theorem 5.3.10 in [11]):

$$\begin{aligned} |\text{iqb}(\tau, \mathbf{n})| &= \frac{1}{1 + \sum_i r_i n_i} \binom{1 + \sum_i r_i n_i}{\sum_i n_i} \binom{\sum_i n_i}{n_1, n_2, \dots, n_m} \\ &= \frac{1}{1 + \sum_i (r_i - 1)n_i} \binom{\sum_i r_i n_i}{\sum_i n_i} \binom{\sum_i n_i}{n_1, n_2, \dots, n_m}. \end{aligned}$$

Next since $\ell = 1$, the set $S_\tau(\mathbf{n}, 0; \ell)$ is just the set of all the balanced words. Hence, $w \in S_\tau(\mathbf{n}, 0; \ell)$ if and only if w contains n_i copies of the letter s_i for each $i > 0$ and $\sum_i (r_i - 1)n_i$ copies of the letter s_0 . Therefore,

$$|S_\tau(\mathbf{n}, 0; \ell)| = \binom{\sum_i r_i n_i}{\sum_i n_i} \binom{\sum_i n_i}{n_1, n_2, \dots, n_m}.$$

Let

$$\begin{aligned} G_\tau(\mathbf{x}) &= \sum_{\mathbf{n} \in \mathbb{N}^m} \frac{1}{1 + \sum_i (r_i - 1)n_i} \binom{\sum_i r_i n_i}{\sum_i n_i} \binom{\sum_i n_i}{n_1, n_2, \dots, n_m} \mathbf{x}^{\mathbf{n}}, \\ H_{\tau, \ell}(\mathbf{x}) &= \sum_{\mathbf{n} \in \mathbb{N}^m} \binom{\sum_i r_i n_i}{\sum_i n_i} \binom{\sum_i n_i}{n_1, n_2, \dots, n_m} \mathbf{x}^{\mathbf{n}}. \end{aligned}$$

Then these are the functions in Proposition 5.9 and Theorem 4.11. Hence,

$$\begin{aligned} \mathcal{S}_{\tau, t; \ell}(\mathbf{x}) &= (G_\tau(\mathbf{x}))^t H_{\tau, \ell}(\mathbf{x}), \text{ and} \\ \mathcal{P}_{\tau, k}(\mathbf{x}) &= (G_\tau(-\mathbf{x}))^{-k - \ell} H_{\tau, \ell}(-\mathbf{x}), \end{aligned}$$

where $\ell = 1$.

Hence, if we let $g(\mathbf{n})$ and $h(\mathbf{n})$ be the coefficients of $\mathbf{x}^{\mathbf{n}}$ in $\log G_\tau(\mathbf{x})$ and $\log H_{\tau, \ell}(\mathbf{x})$ respectively, then as in the proof of Theorem 4.8, we obtain

$$(8.1) \quad \varphi_\tau(\mathbf{n}, k) = (-1)^{\mathbf{n}} (-g(\mathbf{n})k + (h(\mathbf{n}) - \ell g(\mathbf{n}))) = (-1)^{\mathbf{n}} (-g(\mathbf{n})k + (h(\mathbf{n}) - g(\mathbf{n}))).$$

One way to figure out $g(\mathbf{n})$ and $h(\mathbf{n})$ is to use the Lagrange inversion formula. For example, it is known that $z := G_\tau(\mathbf{x})$, the generating function for Lukasiewicz words, satisfies the equation:

$$z = 1 + \sum_{i=1}^m x_i z^{r_i}.$$

Hence, one can use multivariate Lagrange inversion formula to figure out $g(\mathbf{n})$, which is the coefficient of $\mathbf{x}^{\mathbf{n}}$ of $\log(z)$. We demonstrate this approach with a special situation: when $m = 1$ and $\tau = (t_1) = ((\{1\}, r))$, where $r \in \mathbb{P}$.

Suppose $\tau = ((\{1\}, r))$. Since this is a special case of what we've discussed above, we immediately have

$$\begin{aligned}\mathcal{P}_{\tau,k}(x) &= 1 + \sum_{n \in \mathbb{P}} \binom{k - rn + n}{n} x^n = (G_{\tau}(x))^{-k-1} H_{\tau,1}(x), \\ \text{where } G_{\tau}(x) &= \sum_{n \in \mathbb{N}} \frac{1}{1 + (r-1)n} \binom{rn}{n} x^n, \\ H_{\tau,1}(x) &= \sum_{n \in \mathbb{N}} \binom{rn}{n} x^n.\end{aligned}$$

(In fact, words in $\text{iqb}(\tau, n)$ are in one-to-one correspondence to r -ary trees with n internal vertices, whose cardinality is well-known to be $\frac{1}{1 + (r-1)n} \binom{rn}{n}$.)

For convenience, we abbreviate $G_{\tau}(x)$ and $H_{\tau,1}(x)$ to $G(x)$ and $H(x)$ respectively. By Examples 6.2.6 and 6.2.7 in [11], G and H satisfy the following equations:

$$(8.2) \quad G = 1 + xG^r$$

$$(8.3) \quad \begin{aligned}H(x^{r-1}) &= \frac{d}{dx} (xG(x^{r-1})) = G(x^{r-1}) + xG'(x^{r-1})(r-1)x^{r-2} \\ &= G(x^{r-1}) + (r-1)x^{r-1}G'(x^{r-1})\end{aligned}$$

Note that (8.3) implies that

$$(8.4) \quad H = G + (r-1)xG'.$$

By differentiating (8.2), we get

$$G' = G^r + xrG^{r-1}G' \quad \Rightarrow \quad G' = \frac{G^r}{1 - xrG^{r-1}}.$$

Plugging the formula for G' into (8.4), we obtain

$$(8.5) \quad H = G + \frac{x(r-1)G^r}{1 - xrG^{r-1}} = \frac{G - xG^r}{1 - xrG^{r-1}} = \frac{1}{1 - xrG^{r-1}} = \frac{G}{G - xrG^r} = \frac{G}{G - r(G-1)},$$

where both the third and the fifth equalities follow from (8.2).

Let

$$z = G_{\tau}(x) - 1 = \sum_{n \in \mathbb{P}} \frac{1}{1 + (r-1)n} \binom{rn}{n} x^n.$$

Then (8.2) becomes $z = x(z+1)^r$, which is equivalent to

$$z = \left(\frac{x}{(1+x)^r} \right)^{<-1>}.$$

Hence, using the Lagrange inversion formula [11, Corollary 5.4.3], we find that $g(n)$, the coefficient of x^n in $\log(1+z)$, is given by

$$(8.6) \quad g(n) = \frac{1}{n}[t^{n-1}] \left(\frac{1}{1+t} ((1+t)^r)^n \right) = \frac{1}{n} \binom{rn-1}{n-1} = \frac{1}{rn} \binom{rn}{n}.$$

We also rewrite (8.5) using z :

$$H = \frac{G}{G - r(G-1)} = \frac{1+z}{1+z-rz} = \frac{1+z}{1-(r-1)z}.$$

Hence,

$$\log H = \log(1+z) + \log(1-(r-1)z).$$

Therefore, using the Lagrange inversion formula, we find that $h(n)$, the coefficient of x^n in $\log(1+z) - \log(1-(r-1)z)$, is given by

$$(8.7) \quad \begin{aligned} h(n) &= \frac{1}{n}[t^{n-1}] \left(\frac{1}{1+t} + \frac{1}{1-(r-1)t} \right) ((1+t)^r)^n \\ &= g(n) + \frac{1}{n}[t^{n-1}] \left((1+t)^{rn} \sum_{i \geq 0} (r-1)^i t^i \right) \\ &= g(n) + \frac{1}{n} \sum_{i=0}^{n-1} \binom{rn}{i} (r-1)^{n-1-i} \end{aligned}$$

Therefore, applying (8.6) and (8.7) to (8.1), we obtain the following result.

Lemma 8.1. *Suppose $m = 1$ and $\tau = ((\{1\}, r))$. Then*

$$\varphi_\tau(n, k) = \frac{(-1)^{n+1}}{n} \left(\frac{1}{r} \binom{rn}{n} k - \sum_{i=0}^{n-1} \binom{rn}{i} (r-1)^{n-1-i} \right).$$

Note that the above lemma is equivalent to the following lemma.

Lemma 8.2. *For any unknown k , and any positive integer r , the coefficient of x^n in*

$$\log \left(1 + \sum_{n \in \mathbb{P}} \binom{k - rn + n}{n} x^n \right)$$

is given by

$$\frac{(-1)^{n+1}}{n} \left(\frac{1}{r} \binom{rn}{n} k - \sum_{i=0}^{n-1} \binom{rn}{i} (r-1)^{n-1-i} \right).$$

Therefore, we actually are able to compute $\varphi_\tau(n, k)$ for any τ that contains only one type of edges.

Lemma 8.3. *Suppose $m = 1$ and $\tau = ((\{1, 2, \dots, \ell\}, r))$. Then*

$$\varphi_\tau(n, k) = \frac{(-1)^{n+1}}{n} \left(\frac{1}{r\ell} \binom{r\ell n}{n} \left(k\ell + \frac{\ell(\ell-1)}{2} + (\ell-1) \right) - \sum_{i=0}^{n-1} \binom{r\ell n}{i} (r\ell-1)^{n-1-i} \right).$$

Proof. It is clear that we have

$$\lambda_j(G_\tau(n)) = \begin{cases} rn & \text{if } 1 \leq j \leq \ell \\ 0 & \text{if } j > \ell. \end{cases}$$

Hence, $k_{\min}(\tau, n) = rn$. For any $k \geq k_{\min}(\tau, n)$, to create $\text{ext}(G_\tau(n)_{(k)})$, we add $k + (j-1) - rn$ unweighted edges connecting vertices $k + j - 1$ and $k + j$ for any $1 \leq j \leq \ell$. In total, we have

$$\sum_{j=1}^{\ell} (k + (j-1) - rn) = k\ell + \frac{\ell(\ell-1)}{2} - r\ell n$$

unweighted edges between vertices k and $k + \ell$.

One sees that $P(G_\tau(n)_{(k)})$ is the number of ways to place the n edges of weight r between the vertices k and $k + \ell$. Since the other elements that are between vertices k and $k + \ell$ are these $k\ell + \frac{\ell(\ell-1)}{2} - r\ell n$ unweighted edges and $\ell - 1$ vertices: $k + 1, k + 2, \dots, k + \ell - 1$. The order of these two kinds of elements are fixed in any extended ordering. Hence, we conclude that

$$p_\tau(n, k) = P(G_\tau(n)_{(k)}) = \binom{k\ell + \frac{\ell(\ell-1)}{2} - r\ell n + (\ell - 1) + n}{n}.$$

Therefore, $\varphi_\tau(n, k)$ is the coefficient of x^n in the generating function

$$\log \left(1 + \sum_{n \in \mathbb{P}} p_\tau(n, k) x^n \right) = \log \left(1 + \sum_{n \in \mathbb{P}} \binom{k\ell + \frac{\ell(\ell-1)}{2} + (\ell - 1) - r\ell n + n}{n} x^n \right).$$

Then the conclusion follows from Lemma 8.2. \square

Example 8.4. Suppose $\tau = ((\{1, 2\}, r))$. Then $G_\tau(n)$ is the long-edge graph with n edges connecting vertices 0 and 2 and of weight r . By Lemma 8.3, we have

$$\begin{aligned} \varphi_\tau(n, k) &= \frac{(-1)^{n+1}}{n} \left(\frac{1}{2r} \binom{2rn}{n} \left(2k + \frac{2(2-1)}{2} + (2-1) \right) - \sum_{i=0}^{n-1} \binom{2rn}{i} (2r-1)^{n-1-i} \right) \\ &= \frac{(-1)^{n+1}}{n} \left(\frac{1}{r} \binom{2rn}{n} (k+1) - \sum_{i=0}^{n-1} \binom{2rn}{i} (2r-1)^{n-1-i} \right). \end{aligned}$$

Assume further that $r = 1$. Then

$$(8.8) \quad \varphi_\tau(n, k) = \frac{(-1)^{n+1}}{n} \left(\binom{2n}{n} (k+1) - \sum_{i=0}^{n-1} \binom{2n}{i} \right) = \frac{(-1)^{n+1}}{n} \left(\binom{2n}{n} \left(k + \frac{3}{2} \right) - 2^{2n-1} \right),$$

where the second equality follows from the identity

$$\sum_{i=0}^{n-1} \binom{2n}{i} = \frac{1}{2} \left(\sum_{i=0}^{n-1} \binom{2n}{i} + \sum_{i=n+1}^{2n} \binom{2n}{i} \right) = \frac{1}{2} \left(\sum_{i=0}^{2n} \binom{2n}{i} - \binom{2n}{n} \right) = \frac{1}{2} \left(2^{2n} - \binom{2n}{n} \right).$$

In particular, plugging $n = 1$ and 2 in (8.8) we get

$$\varphi_\tau(1, k) = \frac{(-1)^{1+1}}{1} \left(\binom{2}{1} \left(k + \frac{3}{2} \right) - 2^{2-1} \right) = 2k + 1,$$

$$\varphi_\tau(2, k) = \frac{(-1)^{2+1}}{2} \left(\binom{4}{2} \left(k + \frac{3}{2} \right) - 2^{4-1} \right) = -\frac{1}{2}(6k + 1).$$

Note that $G_\tau(1)$ and $G_\tau(2)$ are the second and fifth templates which appeared in Figure 2. The functions we obtain agree with those listed in Figure 2.

REFERENCES

1. F. Block, *Computing node polynomials for plane curves*, Math. Res. Lett. **18** (2011), 621–643.
2. F. Block, S. J. Colley, and G. Kennedy, *Computing Severi degrees with long-edge graphs*, preprint.
3. E. Brugallé and G. Mikhalkin, *Floor decomposition of tropical curves: the planar case*, Proceedings of Gökova Geometry-Topology Conference 2008, Gökova Geometry/Topology Conference (GGT), Gökova, 2009, pp. 64–90.
4. ———, *Enumeration of curves via floor diagrams*, C. R. Math. Acad. Sc. Paris **345** (2007), 329–334.
5. P. Di Francesco and C. Itzykson, *Quantum intersection rings*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 81–148.
6. S. Fomin and G. Mikhalkin, *Labeled floor diagrams for plane curves*, J. Eur. Math. Soc. **12** (2010), 1453–1496.
7. L. Göttsche, *A conjectural generating function for numbers of curves on surfaces*, Comm. Math. Phys. **196** (1998), 523–533.
8. S. Kleiman and V. Shende, *On the Göttsche threshold*, preprint.
9. M. Kool, V. Shende, and R. P. Thomas, *A short proof of the Göttsche conjecture*, Geom. Topol. **15** (2011), 397–406.
10. N. Qviller.
11. R.P. Stanley, *Enumerative combinatorics, vol. 2*, Cambridge Studies in Advanced Mathematics, 62, Cambridge University Press, Cambridge, 1999.
12. ———, *Enumerative combinatorics, vol. 1, 2nd edition*, Cambridge Studies in Advanced Mathematics, 49, Cambridge University Press, Cambridge, 2012.
13. Y. J. Tzeng, *A proof of the Göttsche-Yau-Zaslow formula*, J. Differential Geom. **90** (2012), 439–472.

FU LIU, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CA 95616, USA.
E-mail address: fuliu@math.ucdavis.edu